THE MAX-PLUS FINITE ELEMENT METHOD FOR SOLVING DETERMINISTIC OPTIMAL CONTROL PROBLEMS: BASIC PROPERTIES AND CONVERGENCE ANALYSIS

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Abstract. We introduce a max-plus analogue of the Petrov-Galerkin finite element method to solve finite horizon deterministic optimal control problems. The method relies on a max-plus variational formulation. We show that the error in the sup norm can be bounded from the difference between the value function and its projections on max-plus and min-plus semimodules, when the max-plus analogue of the stiffness matrix is exactly known. In general, the stiffness matrix must be approximated: this requires approximating the operation of the Lax-Oleinik semigroup on finite elements. We consider two approximations relying on the Hamiltonian. We derive a convergence result, in arbitrary dimension, showing that for a class of problems, the error estimate is of order $\delta + \Delta x(\delta)^{-1}$ or $\sqrt{\delta} + \Delta x(\delta)^{-1}$, depending on the choice of the approximation, where δ and Δx are respectively the time and space discretization steps. We compare our method with another max-plus based discretization method previously introduced by Fleming and McEneaney. We give numerical examples in dimension 1 and 2.

1. Introduction

We consider the optimal control problem:

$$\text{maximize} \int_0^T \ell(\mathbf{x}(s), \mathbf{u}(s))\, ds + \phi(\mathbf{x}(T)) \tag{1a}$$

over the set of trajectories $(\mathbf{x}(\cdot), \mathbf{u}(\cdot))$ satisfying

$$\dot{\mathbf{x}}(s) = f(\mathbf{x}(s), \mathbf{u}(s)), \quad \mathbf{x}(s) \in X, \quad \mathbf{u}(s) \in U \ , \tag{1b}$$

for all $0 \le s \le T$ and

$$\mathbf{x}(0) = x \ . \tag{1c}$$

Here, the *state space* X is a subset of $\mathbb{R}^n$, the set of *control values* U is a subset of $\mathbb{R}^m$, the *horizon* $T > 0$ and the *initial condition* $x \in X$ are given, we assume that the map $\mathbf{u}(\cdot)$ is measurable, and that the map $\mathbf{x}(\cdot)$ is absolutely continuous. We also assume that the *instantaneous reward* or *Lagrangian* $\ell : X \times U \to \mathbb{R}$, and the *dynamics* $f : X \times U \to \mathbb{R}^n$, are sufficiently regular maps, and that the *terminal reward* ϕ is a map $X \to \mathbb{R} \cup \{-\infty\}$.

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We are interested in the numerical computation of the *value function* v which associates to any $(x,t) \in X \times [0,T]$ the supremum $v(x,t)$ of $\int_0^t \ell(\mathbf{x}(s),\mathbf{u}(s))\,ds + \phi(\mathbf{x}(t))$, under the constraints (1b), for $0 \leq s \leq t$ and (1c). It is known that, under certain regularity assumptions, v is solution of the Hamilton-Jacobi equation

$$-\frac{\partial v}{\partial t} + H(x, \frac{\partial v}{\partial x}) = 0, \quad (x,t) \in X \times (0,T] \ , \tag{2a}$$

with initial condition:

$$v(x,0) = \phi(x), \quad x \in X \ , \tag{2b}$$

where $H(x,p) = \sup_{u\in U} \ell(x,u) + p \cdot f(x,u)$ is the *Hamiltonian* of the problem (see for instance [Lio82, FS93, Bar94]).

Several techniques have been proposed in the litterature to solve this problem. We mention for example finite difference schemes and the method of the vanishing viscosity [CL84], the anti-diffusive schemes for advection [BZ05], the finite elements approach [GR85] (in the case of the stopping time problem), the so-called discrete dynamic programming method or semi-lagrangian method [CD83], [CDI84], [Fal87], [FF94], [FG99], [CFF04], the Markov chain approximations [BD99]. Other schemes have been obtained by integration from the essentially nonoscillatory (ENO) schemes for the hyperbolic conservation laws (see for instance [OS91]). Recently, max-plus methods have been proposed to solve first-order Hamilton-Jacobi equations [MH98], [MH99], [FM00], [McE02], [McE03], [CM04], [McE04].

Recall that the *max-plus semiring*, $\mathbb{R}_{\max}$, is the set $\mathbb{R} \cup \{-\infty\}$, equipped with the addition $a \oplus b = \max(a,b)$ and the multiplication $a \otimes b = a + b$. In the sequel, let S^t denote the *evolution semigroup* of (2), or Lax-Oleinik semigroup, which associates to any map ϕ the function $v^t := v(\cdot,t)$, where v is the value function of the optimal control problem (1). Maslov [Mas73] observed that the semigroup S^t is *max-plus linear*, meaning that for all maps f, g from X to $\mathbb{R}_{\max}$, and for all $\lambda \in \mathbb{R}_{\max}$, we have

$$S^t(f \oplus g) = S^t f \oplus S^t g \ ,$$
$$S^t(\lambda f) = \lambda S^t f \ ,$$

where $f \oplus g$ denotes the map $x \mapsto f(x) \oplus g(x)$, and λf denotes the map $x \mapsto \lambda \otimes f(x)$. Linear operators over max-plus type semirings have been widely studied, see for instance [CG79, MS92, BCOQ92, KM97, GM01], see also [Fat06].

In [FM00], Fleming and McEneaney introduced a max-plus based discretization method to solve a subclass of Hamilton-Jacobi equations (with a Lagrangian ℓ quadratic with respect to u, and a dynamics f affine with respect to u). They use the max-plus linearity of the semigroup S^t to approximate the value function v^t by a function v_h^t of the form:

$$v_h^t = \sup_{1\leq i\leq p} \{\lambda_i^t + w_i\} \ , \tag{3}$$

where $\{w_i\}_{1\leq i\leq p}$ is a given family of functions (a max-plus "basis") and $\{\lambda_i^t\}_{1\leq i\leq p}$ is a family of scalars (the "coefficients" of v_h^t on the max-plus "basis"), which must be determined. They proposed a discretization scheme in which λ^t is computed inductively by applying a max-plus linear operator to $\lambda^{t-\delta}$, where δ is the time discretization step. Thus, their scheme can be interpreted as the dynamic programming equation of a discrete control problem.

In this paper, we introduce a max-plus analogue of the finite element method, the "MFEM", to solve the deterministic optimal control problem (1). We still look for an approximation v_h^t of the form (3). However, to determine the "coefficients" λ_i^t, we use a max-plus analogue of the notion of variational formulation, which originates from the notion of generalized solution of Hamilton-Jacobi equations of Maslov and Kolokoltsov [KM88], [KM97, Section 3.2]. We choose a family $\{z_j\}_{1\leq j\leq q}$ of test functions and define inductively v_h^t to be the maximal function of the form (3) satisfying

$$\langle v_h^t \mid z_j\rangle \leq \langle S^\delta v_h^{t-\delta} \mid z_j\rangle \qquad \forall 1\leq j\leq q \ , \tag{4}$$

where $\langle\cdot \mid \cdot\rangle$ denotes the max-plus scalar product (see Section 3 for details). We show that the corresponding vector of coefficients λ^t can be obtained by applying to $\lambda^{t-\delta}$ a nonlinear operator, which can be interpreted as the dynamic programming operator of a deterministic zero-sum two players game, with finite action and state spaces. The state space of the game corresponds to the set of finite elements. To each test function corresponds one possible action of the first player, and to each finite element corresponds one possible action of the second player, see Remark 5.

One interest of the MFEM is to provide, as in the case of the classical finite element method, a systematic way to compute error estimates, which can be interpreted geometrically as "projection" errors. In the classical finite element method, orthogonal projectors with respect to the energy norm must be used. In the max-plus case, projectors on semimodules must be used (note that these projectors minimize an additive analogue of Hilbert projective metric [CGQ04]).

We shall see that when the value function is nonsmooth, the space of test functions must be different from the space in which the solution is represented, so that our discretization is indeed a max-plus analogue of the Petrov-Galerkin finite element method. A convenient choice of finite elements and test functions include quadratic functions (also considered by Fleming and McEneaney [FM00]) and norm-like functions, see Section 5.

In the MFEM, we need to compute the value of the max-plus scalar product $\langle z \mid S^\delta w\rangle$ for each finite element w and each test function z. In some special cases, $\langle z \mid S^\delta w\rangle$ can be computed analytically. In general, we need to approximate this scalar product. Here we consider the approximation $S^\delta w(x) = w(x) + \delta H(x, \nabla w(x))$, for $x \in X$, which is also used in [MH99]. Our main result, Theorem 22, provides for the resulting discretization of the value function an error estimate of order $\delta + \Delta x(\delta)^{-1}$, where Δx is the "space discretization step", under classical assumptions on the control problem and the additionnal assumption that the value function v^t is semiconvex for all $t \in [0, T]$. This is comparable with the order obtained in the simplest dicrete dynamic programming method, see [CDI84], [Fal87], [CDF89]. To avoid solving a difficult (nonconvex) optimization problem, we propose a further approximation of the max-plus scalar product $\langle z \mid S^\delta w\rangle$, for which we obtain an error estimate of order $\sqrt{\delta} + \Delta x(\delta)^{-1}$, which is yet comparable to the order of the existing discretization methods [CDI84], [Fal87], [CDF89], [CL84].

Note that the discretization grid need not be regular: in Theorem 22, Δx is defined for an arbitrary grid in term of Voronoi tesselations.

The paper is organised as follows. In Section 2, we recall some basic tools and notions: residuation, semimodules and projection. In Section 3, we present the formulation of the max-plus finite element method. In Section 4 we compare our method with the method proposed by Fleming and McEneaney in [FM00]. In

Section 5, we state an error estimate and we give the main convergence theorem. Finally, in Section 6, we illustrate the method by numerical examples in dimension 1 and 2. Preliminary results of this paper appeared in [AGL04].

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2. Preliminaries on residuation and projections over semimodules

In this section we recall some classical residuation results (see for example [DJLC53], [Bir67], [BJ72], [BCOQ92]), and their application to linear maps on idempotent semimodules (see [LMS01, CGQ04]). We also review some results of [CGQ96, CGQ04] concerning projectors over semimodules. Other results on projectors over semimodules appeared in [Gon96, GM01].

2.1. **Residuation, semimodules, and linear maps.** If $(S, \leq)$ and $(T, \leq)$ are (partially) ordered sets, we say that a map $f : S \to T$ is *monotone* if $s \leq s' \implies f(s) \leq f(s')$. We say that f is *residuated* if there exists a map $f^\sharp : T \to S$ such that

$$f(s) \leq t \iff s \leq f^\sharp(t) \ . \tag{5}$$

The map f is residuated if, and only if, for all $t \in T$, $\{s \in S \mid f(s) \leq t\}$ has a maximum element in S. Then,

$$f^\sharp(t) = \max\{s \in S \mid f(s) \leq t\}, \quad \forall t \in T \ .$$

Moreover, in that case, we have

$$f \circ f^\sharp \circ f = f \text{ and } f^\sharp \circ f \circ f^\sharp = f^\sharp \ . \tag{6}$$

In the sequel, we shall consider situations where S (or T) is equipped with an idempotent monoid law $\oplus$ (*idempotent* means that $a \oplus a = a$). Then the *natural order* on S is defined by $a \leq b \iff a \oplus b = b$. The supremum law for the natural order, which is denoted by $\vee$, coincides with $\oplus$ and the infimum law for the natural order, when it exists, will be denoted by $\wedge$. We say that S is *complete* as a naturally ordered set if any subset of S has a least upper bound for the natural order.

If $\mathcal{K}$ is an idempotent semiring, i.e. a semiring whose addition is idempotent, we say that the semiring $\mathcal{K}$ is *complete* if it is complete as a naturally ordered set, and if the left and right multiplications: $\mathcal{K} \to \mathcal{K}$, $x \mapsto ax$ and $x \mapsto xa$, are residuated. Here and in the sequel, semiring multiplication is denoted by concatenation.

The max-plus semiring, $\mathbb{R}_{\max}$, is an idempotent semiring. It is not complete, but it can be embedded in the complete idempotent semiring $\overline{\mathbb{R}}_{\max}$ obtained by adjoining $+\infty$ to $\mathbb{R}_{\max}$, with the convention that $-\infty$ is absorbing for the multiplication. The map $x \mapsto -x$ from $\overline{\mathbb{R}}$ to itself yields an isomorphism from $\overline{\mathbb{R}}_{\max}$ to the complete idempotent semiring $\overline{\mathbb{R}}_{\min}$, obtained by replacing max by min and by exchanging the roles of $+\infty$ and $-\infty$ in the definition of $\overline{\mathbb{R}}_{\max}$.

Semimodules over semirings are defined like modules over rings, mutatis mutandis, see [LMS01, CGQ04]. When $\mathcal{K}$ is a complete idempotent semiring, we say that a (right) $\mathcal{K}$-semimodule $\mathcal{X}$ is *complete* if it is complete as an idempotent monoid, and if, for all $u \in \mathcal{X}$ and $\lambda \in \mathcal{K}$, the right and left multiplications, $R^{\mathcal{X}}_\lambda : \mathcal{X} \to \mathcal{X}$, $v \mapsto v\lambda$ and $L^{\mathcal{X}}_u : \mathcal{K} \to \mathcal{X}$, $\mu \mapsto u\mu$, are residuated (for the natural order). In a complete semimodule $\mathcal{X}$, we define, for all $u, v \in \mathcal{X}$,

$$u \backslash v \stackrel{\text{def}}{=} (L^{\mathcal{X}}_u)^\sharp(v) = \max\{\lambda \in \mathcal{K} \mid u\lambda \leq v\} \ .$$

We shall use *semimodules of functions*: when X is a set and $\mathcal{K}$ is a complete idempotent semiring, the set of functions $\mathcal{K}^X$ is a complete $\mathcal{K}$-semimodule for the componentwise addition $(u,v) \mapsto u \oplus v$ (defined by $(u \oplus v)(x) = u(x) \oplus v(x)$), and the componentwise multiplication $(\lambda, u) \mapsto u\lambda$ (defined by $(u\lambda)(x) = u(x)\lambda$).

If $\mathcal{K}$ is an idempotent semiring, and if $\mathcal{X}$ and $\mathcal{Y}$ are $\mathcal{K}$-semimodules, we say that a map $A : \mathcal{X} \to \mathcal{Y}$ is *linear*, or is a *linear operator*, if for all $u, v \in \mathcal{X}$ and $\lambda, \mu \in \mathcal{K}$, $A(u\lambda \oplus v\mu) = A(u)\lambda \oplus A(v)\mu$. Then, as in classical algebra, we use the notation Au instead of $A(u)$. When A is residuated and $v \in \mathcal{Y}$, we use the notation $A\backslash v$ or $A^\sharp v$ instead of $A^\sharp(v)$. We denote by $L(\mathcal{X}, \mathcal{Y})$ the set of linear operators from $\mathcal{X}$ to $\mathcal{Y}$. If $\mathcal{K}$ is a complete idempotent semiring, if $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ are complete $\mathcal{K}$-semimodules, and if $A \in L(\mathcal{X}, \mathcal{Y})$ is residuated, then $L(\mathcal{X}, \mathcal{Y})$ and $L(\mathcal{X}, \mathcal{Z})$ are complete $\mathcal{K}$-semimodules and the map $L_A : L(\mathcal{X}, \mathcal{Y}) \to L(\mathcal{X}, \mathcal{Z})$, $B \mapsto A \circ B$, is residuated and we set $A\backslash C := (L_A)^\sharp(C)$, for all $C \in L(\mathcal{X}, \mathcal{Z})$.

If X and Y are two sets, $\mathcal{K}$ is a complete idempotent semiring, and $a \in \mathcal{K}^{X \times Y}$, we construct the linear operator A from $\mathcal{K}^Y$ to $\mathcal{K}^X$ which associates to any $u \in \mathcal{K}^Y$ the function $Au \in \mathcal{K}^X$ such that $Au(x) = \bigvee_{y \in Y} a(x,y)u(y)$. We say that A is the *kernel operator* with *kernel* or *matrix* a. We shall often use the same notation A for the operator and the kernel. As is well known (see for instance [BCOQ92]), the kernel operator A is residuated, and

$$(A\backslash v)(y) = \bigwedge_{x \in X} A(x,y)\backslash v(x) \enspace .$$

In particular, when $\mathcal{K} = \overline{\mathbb{R}}_{\max}$, we have

$$(A\backslash v)(y) = \inf_{x \in X} (-A(x,y) + v(x)) = [-A^*(-v)](y) \enspace , \tag{7}$$

where A^* denotes the *transposed operator* $\mathcal{K}^X \to \mathcal{K}^Y$, which is associated to the kernel $A^*(y,x) = A(x,y)$. (In (7), we use the convention that $+\infty$ is absorbing for addition.)

2.2. **Projectors on semimodules.** Let $\mathcal{K}$ be a complete idempotent semiring and $\mathcal{V}$ denote a *complete subsemimodule* of a complete semimodule $\mathcal{X}$, i.e. a subset of $\mathcal{X}$ that is stable by arbitrary sups and by the action of scalars. We call *canonical projector* on $\mathcal{V}$ the map

$$P_\mathcal{V} : \mathcal{X} \to \mathcal{X}, \quad u \mapsto P_\mathcal{V}(u) = \max\{v \in \mathcal{V} \mid v \leq u\} \enspace . \tag{8}$$

Let W denote a *generating family* of a complete subsemimodule $\mathcal{V}$, which means that any element $v \in \mathcal{V}$ can be written as $v = \bigvee\{w\lambda_w \mid w \in W\}$, for some $\lambda_w \in \mathcal{K}$. It is known that

$$P_\mathcal{V}(u) = \bigvee_{w \in W} w(w\backslash u)$$

(see for instance [CGQ04]). If $B : \mathcal{U} \to \mathcal{X}$ is a residuated linear operator, then when $\mathcal{U}$ and $\mathcal{X}$ are complete semimodules over $\mathcal{K}$, the image $\operatorname{im} B$ of B is a complete subsemimodule of $\mathcal{X}$, and

$$P_{\operatorname{im} B} = B \circ B^\sharp \enspace . \tag{9}$$

The max-plus finite element methods relies on the notion of projection on an image, parallel to a kernel, which was introduced by Cohen, the second author, and Quadrat, in [CGQ96]. The following theorem, of which Proposition 3 below is an immediate corollary, is a variation on the results of [CGQ96, Section 6].

Theorem 1 (Projection on an image parallel to a kernel)**.** *Let $\mathcal{U}$, $\mathcal{X}$ and $\mathcal{Y}$ be complete semimodules over $\mathcal{K}$. Let $B : \mathcal{U} \to \mathcal{X}$ and $C : \mathcal{X} \to \mathcal{Y}$ be two residuated linear operators over $\mathcal{K}$. Let $\Pi_B^C = B \circ (C \circ B)^\sharp \circ C$. We have $\Pi_B^C = \Pi_B \circ \Pi^C$, where $\Pi_B = B \circ B^\sharp$ and $\Pi^C = C^\sharp \circ C$. Moreover, Π_B^C is a projector, meaning that $(\Pi_B^C)^2 = \Pi_B^C$, and for all $x \in \mathcal{X}$:*

$$\Pi_B^C(x) = \max\{y \in \operatorname{im} B \mid Cy \leq Cx\} \enspace .$$

Proof. The first assertion follows from $(C \circ B)^\sharp = B^\sharp \circ C^\sharp$. For the second assertion, we have

$$\begin{aligned}(\Pi_B^C)^2 &= \big(B \circ (C \circ B)^\sharp \circ C\big) \circ \big(B \circ (C \circ B)^\sharp \circ C\big) \\ &= B \circ (C \circ B)^\sharp \circ C \quad (\text{using} \quad (6)) \\ &= \Pi_B^C \enspace .\end{aligned}$$

To prove the last assertion, we use that $\Pi_B = P_{\operatorname{im} B}$ and (5), we deduce:

$$\begin{aligned}\Pi_B^C(x) &= P_{\operatorname{im} B} \circ C^\sharp \circ C(x) \\ &= \max\{y \in \operatorname{im} B \mid y \leq C^\sharp \circ C(x)\} \\ &= \max\{y \in \operatorname{im} B \mid Cy \leq Cx\} \enspace .\end{aligned}$$

□

The results of [CGQ96] characterize the existence and uniqueness, for all $x \in X$, of $y \in \operatorname{im} B$ such that $Cy = Cx$. In that case, $y = \Pi_B^C(x)$.

When $\mathcal{K} = \overline{\mathbb{R}}_{\max}$, and $C : \overline{\mathbb{R}}_{\max}^X \to \overline{\mathbb{R}}_{\max}^Y$ is a kernel operator, $\Pi^C = C^\sharp \circ C$ has an interpretation similar to (9):

$$\Pi^C(v) = C^\sharp \circ C(v) = -P_{\operatorname{im} C^*}(-v) = P^{-\operatorname{im} C^*}(v) \enspace ,$$

where $-\operatorname{im} C^*$ is thought of as a $\overline{\mathbb{R}}_{\min}$-subsemimodule of $\overline{\mathbb{R}}_{\min}^X$ and $P^{\mathcal{V}}$ denotes the projector on a $\overline{\mathbb{R}}_{\min}$-semimodule $\mathcal{V}$, so that,

$$P^{-\operatorname{im} C^*}(v) = \min\{w \in -\operatorname{im} C^* \mid w \geq v\} \enspace ,$$

where $\leq$ denotes here the usual order on $\overline{\mathbb{R}}^X$. When $B : \overline{\mathbb{R}}_{\max}^U \to \overline{\mathbb{R}}_{\max}^X$ is also a kernel operator, we have

$$\Pi_B^C = P_{\operatorname{im} B} \circ P^{-\operatorname{im} C^*} \enspace .$$

This factorization will be instrumental in the geometrical interpretation of the finite element algorithm.

Example 2. We take $U = \{1, \cdots, p\}$, $X = \mathbb{R}$ and $Y = \{1, \cdots, q\}$. Consider the linear operators $B : \overline{\mathbb{R}}_{\max}^U \to \overline{\mathbb{R}}_{\max}^X$ and $C : \overline{\mathbb{R}}_{\max}^X \to \overline{\mathbb{R}}_{\max}^Y$ such that

$$B\lambda(x) = \sup_{1 \leq i \leq p} \{-\frac{c}{2}(x - \hat{x}_i)^2 + \lambda_i\}, \qquad \text{for all } \lambda \in \overline{\mathbb{R}}_{\max}^U \enspace ,$$

and

$$(Cf)_i = \sup_{x \in \mathbb{R}} \{-a|x - \hat{y}_i| + f(x)\}, \qquad \text{for all } f \in \overline{\mathbb{R}}_{\max}^X \enspace .$$

The image of B, $\operatorname{im} B$, is the semimodule generated in the max-plus sense by the functions $x \mapsto -\frac{c}{2}(x - \hat{x}_i)^2$, for $i = 1, \cdots, p$. We have

$$C^\sharp \mu(x) = \inf_{1 \leq i \leq q} \{a|x - \hat{y}_i| + \mu_i\}, \qquad \text{for all } \mu \in \overline{\mathbb{R}}_{\max}^Y \enspace ,$$

and the image of $C^\sharp$, which coincides with $-\operatorname{im} C^*$, is the semimodule generated in the min-plus sense by the functions $x \mapsto a|x - \hat{y}_i|$, for $i = 1, \cdots, q$.
In figure 1(a), we represent a function v and its projection $P^{-\operatorname{im} C^*}(v)$ (in bold). In figure 1(b), we represent (in bold) the projection $P_{\operatorname{im} B}(P^{-\operatorname{im} C^*}(v)) = \Pi_B^C(v)$.

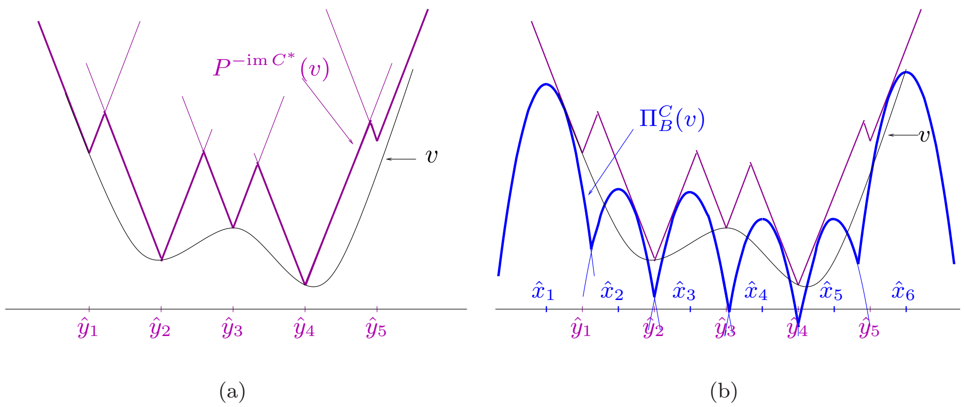

FIGURE 1. Example illustrating max-plus and min-plus projectors

3. The max-plus finite element method

3.1. **Max-plus variational formulation.** We now describe the max-plus finite element method to solve Problem (1). Let $\mathcal{V}$ be a complete semimodule of functions from X to $\overline{\mathbb{R}}_{\max}$. Let $S^t : \mathcal{V} \to \mathcal{V}$ and v^t be defined as in the introduction. Using the semigroup property $S^{t+t'} = S^t \circ S^{t'}$, for $t, t' > 0$, we get:

$$v^{t+\delta} = S^\delta v^t, \quad t = 0, \delta, \cdots, T - \delta \tag{10}$$

with $v^0 = \phi$ and $\delta = \frac{T}{N}$, for some positive integer N. Let $\mathcal{W} \subset \mathcal{V}$ be a complete $\overline{\mathbb{R}}_{\max}$-semimodule of functions from X to $\overline{\mathbb{R}}_{\max}$ such that for all $t \geq 0$, $v^t \in \mathcal{W}$. We choose a "dual" semimodule $\mathcal{Z}$ of "test functions" from X to $\overline{\mathbb{R}}_{\max}$. Recall that the max-plus *scalar product* is defined by

$$\langle u \mid v \rangle = \sup_{x \in X} u(x) + v(x) \ ,$$

for all functions $u, v : X \to \overline{\mathbb{R}}_{\max}$. We replace (10) by:

$$\langle z \mid v^{t+\delta} \rangle = \langle z \mid S^\delta v^t \rangle, \quad \forall z \in \mathcal{Z} \ , \tag{11}$$

for $t = 0, \delta, \ldots, T - \delta$, with $v^\delta, \ldots, v^T \in \mathcal{W}$. Equation (11) can be seen as the analogue of a *variational* or *weak formulation.* Kolokoltsov and Maslov used this formulation in [KM88] and [KM97, Section 3.2] to define a notion of generalized solution of Hamilton-Jacobi equations.

3.2. **Ideal max-plus finite element method.** We consider a semimodule $\mathcal{W}_h \subset \mathcal{W}$ generated by the family $\{w_i\}_{1\leq i\leq p}$. We call *finite elements* the functions w_i. We approximate v^t by $v_h^t \in \mathcal{W}_h$, that is:

$$v^t \simeq v_h^t = \bigvee_{1\leq i\leq p} w_i\lambda_i^t \ ,$$

where $\lambda_i^t \in \overline{\mathbb{R}}_{\max}$. We also consider a semimodule $\mathcal{Z}_h \subset \mathcal{Z}$ with generating family $\{z_j\}_{1\leq j\leq q}$. The functions $z_1, \cdots, z_q$ will act as test functions. We replace (11) by

$$\langle z \mid v_h^{t+\delta}\rangle = \langle z \mid S^\delta v_h^t\rangle, \quad \forall z \in \mathcal{Z}_h \ , \tag{12}$$

for $t = 0, \delta, \ldots, T-\delta$, with $v_h^\delta, \ldots, v_h^T \in \mathcal{W}_h$. The function v_h^0 is a given approximation of ϕ. Since $\mathcal{Z}_h$ is generated by $z_1, \ldots, z_q$, (12) is equivalent to

$$\langle z_j \mid v_h^{t+\delta}\rangle = \langle z_j \mid S^\delta v_h^t\rangle, \quad \forall 1 \leq j \leq q \ , \tag{13}$$

for $t = 0, \delta, \cdots, T-\delta$, with $v_h^t \in \mathcal{W}_h$, $t = 0, \delta, \cdots, T$.

Since Equation (13) need not have a solution, we look for its maximal subsolution, i.e. the maximal solution $v_h^{t+\delta} \in \mathcal{W}_h$ of

$$\langle z_j \mid v_h^{t+\delta}\rangle \quad \leq \quad \langle z_j \mid S^\delta v_h^t\rangle \quad \forall 1 \leq j \leq q \ . \tag{14a}$$

We also take for the approximate value function v_h^0 at time 0 the maximal solution $v_h^0 \in \mathcal{W}_h$ of

$$v_h^0 \leq v^0 \ . \tag{14b}$$

Let us denote by W_h the max-plus linear operator from $\overline{\mathbb{R}}_{\max}^p$ to $\mathcal{W}$ with matrix $W_h = \mathrm{col}(w_i)_{1\leq i\leq p}$, and by Z_h^* the max-plus linear operator from $\mathcal{W}$ to $\overline{\mathbb{R}}_{\max}^q$ whose transposed matrix is $Z_h = \mathrm{col}(z_j)_{1\leq j\leq q}$. This means that $W_h\lambda = \bigvee_{1\leq i\leq p} w_i\lambda_i$ for all $\lambda = (\lambda_i)_{i=1,\ldots,p} \in \overline{\mathbb{R}}_{\max}^p$, and $(Z_h^* v)_j = \langle z_j \mid v\rangle$ for all $v \in \mathcal{W}$ and $j = 1, \ldots, q$. Applying Theorem 1 to $B = W_h$ and $C = Z_h^*$ and noting that $\mathcal{W}_h = \operatorname{im} W_h$, we get:

Corollary 3. *The maximal solution $v_h^{t+\delta} \in \mathcal{W}_h$ of* (14a) *is given by $v_h^{t+\delta} = S_h^\delta(v_h^t)$, where*

$$S_h^\delta = \Pi_{W_h}^{Z_h^*} \circ S^\delta \ .$$

Note that $\Pi_{W_h}^{Z_h^*} = P_{\mathcal{W}_h} \circ P^{-\mathcal{Z}_h}$. The following proposition provides a recursive equation verified by the vector of coordinates of v_h^t.

Proposition 4. *Let $v_h^t \in \mathcal{W}_h$ be the maximal solution of* (14)*, for $t = 0, \delta, \ldots, T$. Then, for every $t = 0, \delta, \ldots, T$, there exists a maximal $\lambda^t \in \mathbb{R}_{\max}^p$ such that $v_h^t = W_h\lambda^t$, $t = 0, \delta, \cdots, T$, which can be determined recursively from*

$$\lambda^{t+\delta} = (Z_h^* W_h)\backslash(Z_h^* S^\delta W_h \lambda^t) \ , \tag{15a}$$

for $t = 0, \cdots, T-\delta$ with the initial condition:

$$\lambda^0 = W_h \backslash \phi \ . \tag{15b}$$

Proof. Since $v_h^t \in \mathcal{W}_h$, $v_h^t = W_h\lambda^t$ for some $\lambda^t \in \overline{\mathbb{R}}_{\max}^p$ and the maximal λ^t satisfying this condition is $\lambda^t = W_h^\sharp(v_h^t)$, for all $t = 0, \delta, \ldots, T$. Since v_h^0 is the maximal solution of (14b), then by (8) and (9), $v_h^0 = P_{\mathcal{W}_h}(\phi) = W_h \circ W_h^\sharp(\phi)$, hence $\lambda^0 =$

$W_h^\sharp \circ W_h \circ W_h^\sharp(\phi) = W_h^\sharp(\phi)$. Let $t = \delta, \ldots, T$. Using Proposition 3, Theorem 1, (6) and the property that $(f \circ g)^\sharp = g^\sharp \circ f^\sharp$ for all residuated maps f and g, we get

$$\begin{aligned}\lambda^{t+\delta} &= W_h^\sharp \circ \Pi_{W_h}^{Z_h^*} \circ S^\delta(W_h \lambda^t) \\ &= W_h^\sharp \circ W_h \circ W_h^\sharp \circ (Z_h^*)^\sharp \circ Z_h^* \circ S^\delta(W_h \lambda^t) \\ &= W_h^\sharp \circ (Z_h^*)^\sharp \circ Z_h^* \circ S^\delta(W_h \lambda^t) \\ &= (Z_h^* W_h)^\sharp (Z_h^* S^\delta W_h \lambda^t) \ .\end{aligned}$$

which yields (15a). □

For $1 \le i \le p$ and $1 \le j \le q$, we define:

$$(M_h)_{ji} = \langle z_j \mid w_i \rangle \tag{16}$$

$$(K_h)_{ji} = \langle z_j \mid S^\delta w_i \rangle \tag{17}$$

$$= \langle (S^*)^\delta z_j \mid w_i \rangle \ , \tag{18}$$

where S^* is the *transposed semigroup* of S, which is the evolution semigroup associated to the optimal control problem (1) in which the sign of the dynamics is changed. The matrices M_h and K_h represent respectively the max-plus linear operators $Z_h^* W_h$ and $Z_h^* S^\delta W_h$. Equation (15a) may be written explicitly, for $1 \le i \le p$, as

$$\lambda_i^{t+\delta} = \min_{1 \le j \le q} \Big(-(M_h)_{ji} + \max_{1 \le k \le p} \big((K_h)_{jk} + \lambda_k^t \big) \Big) \ .$$

Remark 5. This recursion may be interpreted as the dynamic programming equation of a deterministic zero-sum two players game, with finite action and state spaces. Here the state space of the game is the finite set $\{1, \cdots, p\}$ (to each finite element corresponds a state of the game). To each test function corresponds one possible action $j \in \{1, \cdots, q\}$ of the first player, and to each finite element corresponds one possible action $k \in \{1, \cdots, p\}$ of the second player. Given these actions at the state $i \in \{1, \cdots, p\}$, the cost of the first player, which is the reward of the second player, is $-(M_h)_{ji} + (K_h)_{jk}$.

The ideal max-plus finite element method can be summarized as follows:

Algorithm 1 Ideal max-plus finite element method

1: Choose the finite elements $(w_i)_{1 \le i \le p}$ and $(z_j)_{1 \le j \le q}$. Choose the time discretization step $\delta = \frac{T}{N}$,
2: Compute the matrix M_h by (16) and the matrix K_h by (17) or by (18),
3: Compute $\lambda^0 = W_h \backslash \phi$ and $v_h^0 = W_h \lambda^0$.
4: For $t = \delta, 2\delta, \ldots, T$, compute $\lambda^t = M_h \backslash (K_h \lambda^{t-\delta})$ and $v_h^t = W_h \lambda^t$.

Remark 6. Since $v_h^t \in \mathcal{W}_h \ \forall t = 0, \cdots, T$, the dynamics of v_h^tcan be written as a function of the matrices M_h and K_h:

$$v_h^{t+\delta} = W_h \circ M_h^\sharp \circ K_h \circ W_h^\sharp(v_h^t) \ . \tag{19}$$

3.3. **Effective max-plus finite element method.** In order to implement the max-plus finite element method, we must specify how to compute the entries of the matrices M_h and K_h in (16) and (17) or (18).
Computing M_h from (16) is an optimization problem, whose objective function is concave for natural choices of finite elements and test functions (see Section 5 below). This problem may be solved by standard optimization algorithms. Evaluating every scalar product $\langle z \mid S^\delta w\rangle$ leads to a new optimal control problem since

$$\langle z \mid S^\delta w\rangle = \max z(\mathbf{x}(0)) + \int_0^\delta \ell(\mathbf{x}(s), \mathbf{u}(s))ds + w(\mathbf{x}(\delta)) \ ,$$

where the maximum is taken over the set of trajectories $\big(\mathbf{x}(\cdot), \mathbf{u}(\cdot)\big)$ satisfying (1b). This problem is simpler to approximate than Problem (1), because the horizon δ is small, and the functions z and w have a regularizing effect.
We first discuss the approximation of $S^\delta w$ for every finite element w. The Hamilton-Jacobi equation (2a) suggests to approximate $S^\delta w$ by the function $[S^\delta w]_H$ such that

$$[S^\delta w]_H(x) = w(x) + \delta H(x, \nabla w(x)), \quad \text{for all } x \in X \ . \tag{20}$$

Let $[S^\delta W_h]_H$ denote the max-plus linear operator from $\overline{\mathbb{R}}^p_{\max}$ to $\mathcal{W}$ with matrix $[S^\delta W_h]_H = \mathrm{col}([S^\delta w_i]_H)_{1\leq i\leq p}$, which means that

$$[S^\delta W_h]_H \lambda = \bigvee_{1\leq i\leq p} [S^\delta w_i]_H \lambda_i$$

for all $\lambda = (\lambda_i)_{1\leq i\leq p} \in \mathbb{R}^p_{\max}$. The above approximation of $S^\delta w$ yields an approximation of the matrix K_h by the matrix $K_{H,h} := Z_h^*[S^\delta W_h]_H$, whose entries are given, for $1 \leq i \leq p$ and $1 \leq j \leq q$, by:

$$(K_{H,h})_{ji} \quad = \quad \sup_{x\in X}(z_j(x) + w_i(x) + \delta H(x, \nabla w_i(x))) \ . \tag{21}$$

Thus, computing $K_{H,h}$ requires to solve an optimization problem, which is nothing but a perturbation of the optimization problem associated to the computation of M. We may exploit this observation by replacing $K_{H,h}$ by the matrix $\tilde{K}_{H,h}$ with entries

$$(\tilde{K}_{H,h})_{ji} \quad = \quad \langle z_j \mid w_i\rangle + \delta \sup_{x\in \arg\max\{z_j+w_i\}} H(x, \nabla w_i(x)) \ , \tag{22}$$

for $1 \leq i \leq p$ and $1 \leq j \leq q$. Here, $\arg\max\{z_j + w_i\}$ denotes the set of x such that $z_j(x) + w_i(x) = \langle z_j \mid w_i\rangle$. When this set has only one element, (22) yields a convenient approximation of K_h.

Of course, w_i must be differentiable for the approximation (20) to make sense. When w_i is non-differentiable, but z_j is differentiable, the dual formula (18) suggests to approximate $(K_h)_{ji}$ by

$$\sup_{x\in X}(z_j(x) + \delta H(x, -\nabla z_j(x)) + w_i(x)) \ .$$

We may also use the dual formula of (22), where $\nabla w_i(x)$ is replaced by $-\nabla z_j(x)$.

4. Comparison with the method of Fleming and McEneaney

Fleming and McEneaney proposed a max-plus based method [FM00], which also uses a space $\mathcal{W}_h$ generated by finite elements, $w_1, \dots, w_p$, together with the linear formulation (10). Their method approaches the value function at time t, v^t, by $W_h\mu^t$, where $W_h = \mathrm{col}(w_i)_{1\le i\le p}$ as above, and μ^t is defined inductively by

$$\mu^0 = W_h \backslash \phi \tag{23a}$$

$$\mu^{t+\delta} = \big(W_h\backslash(S^\delta W_h)\big)\mu^t \ , \tag{23b}$$

for $t = 0, \delta, \dots, T-\delta$. This can be compared with the limit case of our finite element method, in which the space of test functions $\mathcal{Z}_h$ is the set of all functions. This limit case corresponds to replacing Z_h^* by the identity operator in (15a), so that

$$\lambda^{t+\delta} = W_h\backslash(S^\delta W_h\lambda^t) \ . \tag{24}$$

Proposition 7. *Let (μ^t) be the sequence of vectors defined by the algorithm of Fleming and McEneaney,* (23)*; let (λ^t) be the sequence of vectors defined by the max-plus finite element method, in the limit case* (24)*; and let v^t denote the value function at time t. Then,*

$$W_h\mu^t \le W_h\lambda^t \le v^t \ , \quad \text{for } t = 0, \delta, \dots, T \ . \tag{25}$$

Proof. We first prove that $W_h\lambda^t \le v^t$ for $t = 0, \delta, \dots, T$. This can be proved by induction. For $t = 0$ we have $W_h\lambda^0 \le v^0$ by (14b). We assume that $W_h\lambda^t \le v^t$. Using (24), we have

$$\begin{aligned} W_h\lambda^{t+\delta} &= W_hW_h^\sharp S^\delta(W_h\lambda^t) \\ &= \Pi_{W_h}\big(S^\delta(W_h\lambda^t)\big) \ . \end{aligned}$$

Using the monotonicity of the semigroup S^δ, we obtain

$$\begin{aligned} W_h\lambda^{t+\delta} &\le \Pi_{W_h}\big(S^\delta v^t\big) \\ &\le S^\delta v^t \\ &= v^{t+\delta} \ . \end{aligned}$$

The second inequality is also proved by induction. For $t = 0$, we have $\mu^0 = \lambda^0 = W_h\backslash\Phi$. Suppose that $\mu^t \le \lambda^t$. By definition of $W_h\backslash\big(S^\delta W_h\big)$, we have

$$W_h\big(W_h\backslash S^\delta W_h\big) \le S^\delta W_h \ ,$$

hence

$$\begin{aligned} W_h\mu^{t+\delta} &= W_h\big(W_h\backslash S^\delta W_h\big)\mu^t \\ &\le \big(S^\delta W_h\big)\mu^t \\ &\le S^\delta W_h\lambda^t \ . \end{aligned}$$

Since

$$\begin{aligned} \lambda^{t+\delta} &= W_h\backslash\big(S^\delta W_h\lambda^t\big) \\ &= \max\{\lambda\in\overline{\mathbb{R}}^p_{\max} \mid W_h\lambda \le S^\delta W_h\lambda^t\} \ , \end{aligned}$$

we get that $\mu^{t+\delta} \le \lambda^{t+\delta}$. Then $\mu^t \le \lambda^t$ for $t = 0, \delta, ..., T$. Since W_h is monotone, we deduce (25). □

An approximation of (23b) using formulae of the same type as (20) is also discussed in [MH99].

5. Error analysis

5.1. **General error estimates.** In the sequel we denote by $\|v\|_\infty = \sup_{i\in I} |v(i)| \in \mathbb{R}\cup\{+\infty\}$ the sup-norm of any function $v : I \to \mathbb{R}$. We also use the same notation $\|v\|_\infty = \sup_{i\in I} |v_i|$ for a vector $v = (v_i)_{i\in I}$. For any two sets I and J, a map $\Phi : \mathbb{R}^I \to \mathbb{R}^J$ is said monotone and homogeneous if it is monotone for the natural order and if for all $u \in \mathbb{R}^I$ and $\lambda \in \mathbb{R}$, $\Phi(u + \lambda) = \Phi(u) + \lambda$ with $(u + \lambda)(i) = u(i) + \lambda$. Monotone homogeneous maps are nonexpansive for the sup-norm: $\|\Phi(u) - \Phi(v)\|_\infty \leq \|u - v\|_\infty$, see [CT80]. In particular, max-plus or min-plus linear operators are non-expansive for the sup-norm. This property will be frequently used in the sequel. In order to simplify notations, we denote $\bar\tau_\delta = \{0, \delta, \cdots , T\}$, $\tau_\delta^+ = \bar\tau_\delta\backslash\{0\}$ and $\tau_\delta^- = \bar\tau_\delta\backslash\{T\}$.

Remark 8. To establish the main result of the paper (Theorem 22 below), we shall need only to take the norm of finite valued functions. However, we wish to emphasize that all the computations that follow are valid for functions with values in $\overline{\mathbb{R}}$ if one replaces every occurence of a term of the form $\|u - v\|_\infty$ by $d_\infty(u, v) = \inf\{\lambda \geq 0 \mid -\lambda+v \leq u \leq \lambda+v\}$. Observe that $d_\infty(u, v)$ is a semidistance and that $d_\infty(u, v) = \|u - v\|_\infty$, if $u - v$ takes finite values. Observe also that if a map $\Phi : \overline{\mathbb{R}}^I \to \overline{\mathbb{R}}^J$ is monotone and homogeneous, $d_\infty(\Phi(u), \Phi(v)) \leq d_\infty(u, v)$, for all $u, v \in \overline{\mathbb{R}}^I$.

The following lemma shows that the error of the ideal max-plus finite element method is controlled by the projection errors $\|\Pi^{Z_h^*}_{W_h}(v^t) - v^t\|_\infty$. This lemma may be thought of as an analogue of Cea's lemma in the classical analysis of the errors of the finite element method. Projectors over semimodules in the MFEM correspond to orthogonal projectors in the classical finite element method.

Lemma 9. *For $t \in \bar\tau_\delta$, let v^t be the value function at time t, and v_h^t be its approximation given by the ideal max-plus finite element method. We have*

$$\|v_h^T - v^T\|_\infty \leq \|\Pi_{W_h}(v^0) - v^0\|_\infty + \sum_{t\in\tau_\delta^+} \|\Pi^{Z_h^*}_{W_h}(v^t) - v^t\|_\infty \quad . \tag{26}$$

Proof. For all $t \in \tau_\delta^-$, we have

$$\begin{aligned}\|v_h^{t+\delta} - v^{t+\delta}\|_\infty &\leq \|v_h^{t+\delta} - S_h^\delta(v^t)\|_\infty + \|S_h^\delta(v^t) - v^{t+\delta}\|_\infty \\ &\leq \|S_h^\delta(v_h^t) - S_h^\delta(v^t)\|_\infty + \|\Pi^{Z_h^*}_{W_h} \circ S^\delta(v^t) - v^{t+\delta}\|_\infty \quad .\end{aligned}$$

Since S_h^δ is a non-expansive operator, we deduce

$$\|v_h^{t+\delta} - v^{t+\delta}\|_\infty \leq \|v_h^t - v^t\|_\infty + \|\Pi^{Z_h^*}_{W_h}(v^{t+\delta}) - v^{t+\delta}\|_\infty \quad .$$

The result is obtained by induction on t, using the fact that $v_h^0 = P_{\mathcal{W}_h}(v^0) = \Pi_{W_h}(v^0)$. □

To obtain an error estimate, we need to bound $\|\Pi^{Z_h^*}_{W_h}(v^t) - v^t\|_\infty$ for all $t \in \tau_\delta^+$. Since $\Pi^{Z_h^*}_{W_h} = \Pi_{W_h} \circ \Pi^{Z_h^*}$, we have

$$\begin{aligned}\|\Pi^{Z_h^*}_{W_h}(v^t) - v^t\|_\infty &= \|\Pi_{W_h} \circ \Pi^{Z_h^*}(v^t) - v^t\|_\infty \\ &\leq \|\Pi_{W_h} \circ \Pi^{Z_h^*}(v^t) - \Pi_{W_h}(v^t)\|_\infty + \|\Pi_{W_h}(v^t) - v^t\|_\infty \quad ,\end{aligned}$$

and since Π_{W_h} is a non-expansive operator, we get

$$\|\Pi_{W_h}^{Z_h^*}(v^t) - v^t\|_\infty \leq \|\Pi^{Z_h^*}(v^t) - v^t\|_\infty + \|\Pi_{W_h}(v^t) - v^t\|_\infty \ . \tag{27}$$

Using this inequality together with Lemma 9, we deduce the following corollary.

Corollary 10. *For $t \in \bar{\tau}_\delta$, let v^t be the value function at time t, and v_h^t be its approximation given by the ideal max-plus finite element method. We have*

$$\|v_h^T - v^T\|_\infty \leq (1 + \frac{T}{\delta})\Big(\sup_{t\in\bar{\tau}_\delta}(\|\Pi^{Z_h^*}(v^t) - v^t\|_\infty + \|\Pi_{W_h}(v^t) - v^t\|_\infty)\Big) \ .$$

The following general lemma shows that the error of the effective finite element method is controlled by the projection errors and the errors resulting from the approximation of the matrix K_h by a matrix $\tilde{K}_h$.

Lemma 11. *For $t \in \bar{\tau}_\delta$, let v^t be the value function at time t, and v_h^t be its approximation given by the effective max-plus finite element method, where K_h is approximated by $\tilde{K}_h$. We have*

$$\|v_h^T - v^T\|_\infty \leq (1 + \frac{T}{\delta})\Big(\sup_{t\in\bar{\tau}_\delta}(\|\Pi^{Z_h^*}(v^t) - v^t\|_\infty + \|\Pi_{W_h}(v^t) - v^t\|_\infty) + \|\tilde{K}_h - K_h\|_\infty\Big) \ .$$

Proof. Since v_h^t is computed with the approximation $\tilde{K}_h$ of K_h, we have $v_h^t = W_h\lambda^t$, $t \in \bar{\tau}_\delta$, with

$$\lambda^{t+\delta} = M_h^\sharp \circ (\tilde{K}_h\lambda^t) = W_h^\sharp \circ (Z_h^*)^\sharp \circ (\tilde{K}_h\lambda^t) \ .$$

We have

$$\begin{aligned}
\|v_h^{t+\delta} - v^{t+\delta}\|_\infty &\leq \|v_h^{t+\delta} - S_h^\delta v_h^t\|_\infty + \|S_h^\delta v_h^t - S_h^\delta v^t\|_\infty + \|S_h^\delta v^t - v^{t+\delta}\|_\infty \\
&\leq \|\Pi_{W_h} \circ (Z_h^*)^\sharp \circ (\tilde{K}_h\lambda^t) - \Pi_{W_h} \circ (Z_h^*)^\sharp \circ Z_h^* \circ S^\delta W_h\lambda^t\|_\infty \\
&\qquad + \|v_h^t - v^t\|_\infty + \|\Pi_{W_h}^{Z_h^*}(v^{t+\delta}) - v^{t+\delta}\|_\infty \\
&\leq \|\tilde{K}_h\lambda^t - K_h\lambda^t\|_\infty + \|v_h^t - v^t\|_\infty + \|\Pi_{W_h}^{Z_h^*}(v^{t+\delta}) - v^{t+\delta}\|_\infty \\
&\leq \max_{\substack{1\leq j\leq q\\ 1\leq i\leq p}} |(\tilde{K}_h)_{ji} - (K_h)_{ji}| + \|v_h^t - v^t\|_\infty + \|\Pi_{W_h}^{Z_h^*}(v^{t+\delta}) - v^{t+\delta}\|_\infty \ .
\end{aligned}$$

We deduce that

$$\|v_h^T - v^T\|_\infty \leq \|\Pi_{W_h}(v^0) - v^0\|_\infty + \sum_{t\in\tau_\delta^+}\Big(\|\Pi_{W_h}^{Z_h^*}(v^t) - v^t\|_\infty + \|\tilde{K}_h - K_h\|_\infty\Big) \ ,$$

and so

$$\|v_h^T - v^T\|_\infty \leq (1 + \frac{T}{\delta})\Big(\sup_{t\in\bar{\tau}_\delta}(\|\Pi^{Z_h^*}(v^t) - v^t\|_\infty + \|\Pi_{W_h}(v^t) - v^t\|_\infty) + \|\tilde{K}_h - K_h\|_\infty\Big) \ .$$

□

Corollary 12. *For $t \in \bar{\tau}_\delta$, let v^t be the value function at time t, and v_h^t be its approximation given by the effective max-plus finite element method, implemented*

with the approximation $K_{H,h}$ of K_h, given by (21)*. We have*

$$\|v_h^T - v^T\|_\infty \leq (1+\frac{T}{\delta})\Big(\sup_{t\in\bar\tau_\delta}(\|\Pi^{Z_h^*}(v^t)-v^t\|_\infty + \|\Pi_{W_h}(v^t)-v^t\|_\infty) + \max_{1\leq i\leq p}\|[S^\delta w_i]_H - S^\delta w_i\|_\infty\Big) \ .$$

Proof. Using the same technique as in the precedent lemma and using that $K_{H,h} = Z_h^*[S^\delta W_h]_H$ and $K_h = Z_h^* S^\delta W_h$ we have

$$\begin{aligned}\|K_{H,h}-K_h\|_\infty &\leq \|[S^\delta W_h]_H - S^\delta W_h\|_\infty\\ &= \max_{1\leq i\leq p}\|[S^\delta w_i]_H - S^\delta w_i\|_\infty \ ,\end{aligned} \tag{28}$$

which ends the proof. □

Corollary 13. *For $t\in\bar\tau_\delta$, let v^t be the value function at time t, and v_h^t be its approximation given by the effective max-plus finite element method, implemented with the approximation $\tilde K_{H,h}$ of K_h, given by* (22)*. We have*

$$\|v_h^T - v^T\|_\infty \leq (1+\frac{T}{\delta})\Big(\sup_{t\in\bar\tau_\delta}(\|\Pi^{Z_h^*}v^t-v^t\|_\infty + \|\Pi_{W_h}v^t-v^t\|_\infty) + \max_{1\leq i\leq p}\|[S^\delta w_i]_H - S^\delta w_i\|_\infty + \|\tilde K_{H,h}-K_{H,h}\|_\infty\Big) \ .$$

Proof. We use Lemma 11, together with Equation (28) and

$$\|\tilde K_{H,h}-K_h\|_\infty \leq \|\tilde K_{H,h}-K_{H,h}\|_\infty + \|K_{H,h}-K_h\|_\infty \ .$$

□

5.2. **Projection errors.** In this section, we estimate the projection errors resulting from different choices of finite elements. Recall that a function f is *c-semiconvex* if $f(x)+\frac{c}{2}\|x\|_2^2$, where $\|\cdot\|_2$ is the standard euclidean norm of $\mathbb{R}^n$, is convex. A function f is *c-semiconcave* if $-f$ is c-semiconvex. Spaces of semiconvex functions were intensively used in the max-plus based approximation method of Fleming and McEneaney [FM00], see also [MH98], [MH99], [McE02], [McE03], [McE04], [Fal87], [CDI84], [CDF89].

We shall use the following finite elements.

Definition 14 (P_1 finite elements)**.** We call P_1 *finite element* or *Lipschitz finite element* centered at point $\hat x\in X$, with constant $a>0$, the function $w(x) = -a\|x-\hat x\|_1$ where $\|x\|_1 = \sum_{i=1}^n |x_i|$ is the l^1-norm of $\mathbb{R}^n$.

The family of Lipschitz finite element of constant a generates, in the max-plus sense, the semimodule of Lipschitz continuous functions from X to $\bar{\mathbb{R}}$ of Lipschitz constant a with respect to $\|\cdot\|_1$.

Definition 15 (P_2 finite elements)**.** We call P_2 *finite element* or *quadratic finite element* centered at point $\hat x\in X$, with Hessian $c>0$, the function $w(x) = -\frac{c}{2}\|x-\hat x\|_2^2$.

When $X=\mathbb{R}^n$, the family of quadratic finite elements with Hessian c generates, in the max-plus sense, the semi-module of lower-semicontinuous c-semiconvex functions with values in $\bar{\mathbb{R}}$.

Notations. Let Y be a subset of $\mathbb{R}^n$ and f be a function from Y to $\overline{\mathbb{R}}$. We will

denote by Conv$Y$ the convex hull of Y, ri$Y$ the relative interior of Y, dom$f$ the effective domain of f and $\partial f(x)$ the subdifferential of f at $x \in \mathrm{dom} f$.
When C is a nonempty convex subset of $\mathbb{R}^n$ and $c > 0$, a fonction is said to be *c-strongly convex* on C if and only if $f - \frac{1}{2}c\|\cdot\|_2^2$ is convex on C. A function f is *c-strongly concave* on C if $-f$ is c-strongly convex on C.

Let P be a finite subset of $\mathbb{R}^n$. The *Voronoi cell* of a point $p \in P$ is defined by

$$V(p) = \{x \in \mathbb{R}^n \mid \|x-p\|_2 \leq \|x-q\|_2, \forall q \in P\}.$$

The family $\{V(p)\}_{p\in P}$ constitutes a subdivison of $\mathbb{R}^n$, which is called a Voronoi tesselation (see [SU00] for an introduction to Voronoi tesselations). We define *the restriction of* $V(p)$ *to* X to be:

$$V_X(p) = V(p) \cap X.$$

We define $\rho_X(P)$ to be the *maximal radius* of the restriction to X of the Voronoi cells of the points of P:

$$\rho_X(P) := \sup_{p\in P} \sup_{x \in V_X(p)} \|x-p\|_2.$$

Observe that

$$\rho_X(P) := \sup_{x\in X} \inf_{p \in P} \|x-p\|_2.$$

The previous definitions are illustrated in Figure 2. The set X is in light gray, $P = \{p_1, \cdots, p_{10}\}$, $V_X(P_9)$ is in dark gray and $\rho_X(P)$ is indicated by a bidirectional arrow.

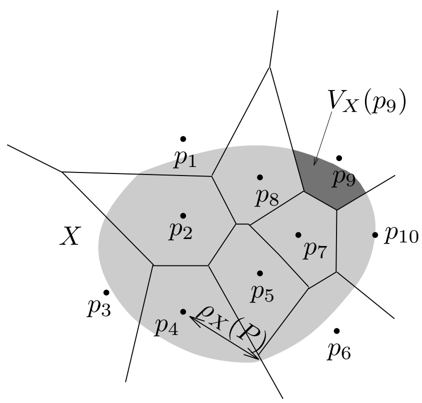

FIGURE 2. Voronoi tesselation

The next two lemmas bound the projection error in term of the radius of Voronoi cells.

Lemma 16 (Primal projection error)**.** *Let X be a compact convex subset of $\mathbb{R}^n$. Let $v : X \to \mathbb{R}$ be c-semiconvex and Lipschitz continuous function with Lipschitz constant L_v with respect to the euclidean norm. Let $v_c(x) = v(x) + \frac{c}{2}\|x\|_2^2$. Let $\hat{X} = X + \mathrm{B}_2(0, \frac{L_v}{c})$, let $\hat{X}_h$ be a finite subset of $\mathbb{R}^n$ and let $\mathcal{W}_h$ denote the complete subsemimodule of $\overline{\mathbb{R}}_{\max}^X$ generated by the family $(w_{\hat{x}_h})_{\hat{x}_h \in \hat{X}_h}$ where $w_{\hat{x}_h}(x) = -\frac{c}{2}\|x - \hat{x}_h\|_2^2$. Then*

$$\|v - P_{\mathcal{W}_h} v\|_\infty \leq c \,\mathrm{diam}\, X \rho_{\hat{X}}(\hat{X}_h)$$

Proof. Let $\mathcal{W}$ denote the complete subsemimodule of $\overline{\mathbb{R}}_{\max}^X$ generated by the family $(w_{\hat{x}})_{\hat{x}\in\hat{X}}$. We will first prove that for all $x \in X$, $P_{\mathcal{W}}v(x) = v(x)$. It is obvious that $\forall x \in X$, $v(x) \geq P_{\mathcal{W}}v(x)$. Using that $P_{\mathcal{W}} = W \circ W^\sharp$, with $W = \mathrm{col}(w_{\hat{x}})_{\hat{x}\in\hat{X}}$, we obtain

$$\begin{aligned} P_{\mathcal{W}}v(x) &= \sup_{\hat{x}\in\hat{X}} \Big(-\frac{c}{2}\|x-\hat{x}\|_2^2 + \inf_{y\in X} \big(\frac{c}{2}\|y-\hat{x}\|_2^2 + v(y)\big)\Big) \\ &= \sup_{\hat{x}\in\hat{X}} \Big(-\frac{c}{2}\|x\|_2^2 + c\hat{x}\cdot x - \sup_{y\in X} \big(c\hat{x}\cdot y - v_c(y)\big)\Big) \\ &= -\frac{c}{2}\|x\|_2^2 + \sup_{\hat{x}\in\hat{X}} \big(c\hat{x}\cdot x - v_c^*(c\hat{x})\big) \ , \end{aligned}$$

where v_c^* denotes the Fenchel transform of v_c. Since v_c is l.s.c., convex and proper, we have for all $x \in X$

$$v_c(x) = v_c^{**}(x) = \sup_{\theta\in\mathbb{R}^n} \big(\theta\cdot x - v_c^*(\theta)\big) \ . \tag{29}$$

Using Theorem 23.4 of [Roc70], for all $x \in \mathrm{ri}(\mathrm{dom} v_c)$, the subdifferential of v_c at x, $\partial v_c(x) = \{\theta \in \mathbb{R}^n \mid v_c(y) - v_c(x) \geq \theta\cdot(y-x), \forall y \in X\}$, is non-empty. Then $\theta \in \partial v_c(x)$ if and only if $v_c^*(\theta) = \theta\cdot x - v_c(x)$ and consequently, the supremum of (29) is attained for all elements θ of $\partial v_c(x)$.
Set $q(x) = \frac{c}{2}\|x\|_2^2$. Using the fact that $q(y) - q(x) = q'(x)\cdot(y-x) + O(\|y-x\|_2^2)$ and that v is Lipschitz continuous with Lipschitz constant L_v, we obtain $\partial v_c(x) \subset \mathrm{B}_2(cx, L_v)$ for all $x \in \mathrm{ri}X$. Therefore, for all $x \in \mathrm{ri}X$,

$$v_c(x) = \sup_{\hat{x}\in\hat{X}} \big(c\hat{x}\cdot x - v_c^*(c\hat{x})\big) \ . \tag{30}$$

By continuity in the members of Equation (30), we have the equality for all $x \in X$, and so

$$\begin{aligned} P_{\mathcal{W}}v(x) &= -\frac{c}{2}\|x\|_2^2 + \sup_{\hat{x}\in\hat{X}} \big(c\hat{x}\cdot x - v_c^*(c\hat{x})\big) \\ &= -\frac{c}{2}\|x\|_2^2 + v_c(x) \\ &= v(x) \ , \end{aligned}$$

for all $x \in X$.
Now, fix $x \in X$. For $\hat{x} \in \hat{X}$, we set $\varphi(\hat{x}) = c\hat{x}\cdot x - v_c^*(c\hat{x})$. Since $P_{\mathcal{W}_h}v \leq P_{\mathcal{W}}v \leq v$, we have for all $x \in X$

$$\begin{aligned} 0 \leq v(x) - P_{\mathcal{W}_h}v(x) &= P_{\mathcal{W}}v(x) - P_{\mathcal{W}_h}v(x) \\ &= \sup_{\hat{x}\in\hat{X}} \varphi(\hat{x}) - \sup_{\hat{x}_h\in\hat{X}_h} \varphi(\hat{x}_h) \\ &= \sup_{\hat{x}\in\hat{X}} \inf_{\hat{x}_h\in\hat{X}_h} \varphi(\hat{x}) - \varphi(\hat{x}_h) \ . \end{aligned}$$

We have $\partial(-\varphi)(\hat{x}) = -cx + c\partial v_c^*(c\hat{x})$. Since $\partial v_c^* \subset X$, we have $\partial(-\varphi)(\hat{x}) \subset c(X - x) \subset \mathrm{B}_2(0, c\,\mathrm{diam}\,X)$. Hence, φ is Lipschitz continuous with Lipschitz constant $L_\varphi = c\,\mathrm{diam}\,X$. Then for all $x \in X$

$$\begin{aligned} v(x) - P_{\mathcal{W}_h}v(x) &\leq \sup_{\hat{x}\in\hat{X}} \inf_{\hat{x}_h\in\hat{X}_h} L_\varphi\|\hat{x} - \hat{x}_h\|_2 \\ &= c\,\mathrm{diam}\,X\rho_{\hat{X}}(\hat{X}_h) \ . \end{aligned}$$

$\square$

Lemma 17 (Dual projection error). *Let X be a bounded subset of $\mathbb{R}^n$ and $\hat{X}$ a finite subset of $\mathbb{R}^n$. Let $v : X \to \mathbb{R}$ be a given Lipschitz continuous function with Lipschitz constant L_v with respect to the euclidean norm. Let $\mathcal{Z}_{\hat{X}}$ denote the complete semimodule of $\overline{\mathbb{R}}_{\max}^X$ generated by the P_1 finite elements $(z_{\hat{x}})_{\hat{x}\in\hat{X}}$ centered at the points of $\hat{X}$ with constant $a \geq L_v$. Then*

$$\|v - P^{-(\mathcal{Z}_{\hat{X}})}v\|_\infty \leq n(a + L_v)\rho_X(\hat{X}).$$

Proof. It is clear that $P^{-(\mathcal{Z}_{\hat{X}})}v \geq v$ and using that $P^{-(\mathcal{Z}_{\hat{X}})} = (Z^*)^\sharp \circ Z^*$, with $Z = \mathrm{col}(z_{\hat{x}})_{\hat{x}\in\hat{X}}$, we obtain

$$P^{-(\mathcal{Z}_{\hat{X}})}v(x) - v(x) = \inf_{\hat{x}\in\hat{X}} \Big(a\|x-\hat{x}\|_1 + \sup_{y\in X} \big(-a\|y-\hat{x}\|_1 + v(y) - v(x) \big) \Big) \ ,$$

for all $x \in X$. Since v is L_v-Lipschitz continuous, we have

$$\begin{aligned}
P^{-(\mathcal{Z}_{\hat{X}})}v(x) - v(x) &\leq \inf_{\hat{x}\in\hat{X}} \Big(a\|x-\hat{x}\|_1 + \sup_{y\in X} \big(-a\|y-\hat{x}\|_1 + L_v\|y-x\|_2 \big) \Big) \\
&\leq \inf_{\hat{x}\in\hat{X}} \Big(a\|x-\hat{x}\|_1 + \sup_{y\in X} \big(-a\|y-\hat{x}\|_1 + L_v\|y-x\|_1 \big) \Big) \\
&\leq \inf_{\hat{x}\in\hat{X}} \Big(a\|x-\hat{x}\|_1 + \sup_{y\in X} \big(-a\|y-\hat{x}\|_1 + L_v\|y-\hat{x}\|_1 \\
&\qquad\qquad + L_v\|x-\hat{x}\|_1 \big) \Big) \\
&= \inf_{\hat{x}\in\hat{X}} \Big((a+L_v)\|x-\hat{x}\|_1 + \sup_{y\in X} (L_v - a)\|y-\hat{x}\|_1 \Big) \ .
\end{aligned}$$

Since $a \geq L_v$, we deduce

$$P^{-(\mathcal{Z}_{\hat{X}})}v(x) - v(x) \leq (a+L_v) \sup_{x\in X} \inf_{\hat{x}\in\hat{X}} \|x-\hat{x}\|_1 \leq n(a+L_v)\rho_X(\hat{X}) \ .$$

□

5.3. **The approximation errors.** To state an error estimate, we make the following standard assumptions (see [Bar94] for instance):

- $(H1)$ $f : X \times U \to \mathbb{R}^n$ is bounded and Lipschitz continuous with respect to x, meaning that there exist $L_f > 0$ and $M_f > 0$ such that

$$\begin{aligned}
\|f(x,u) - f(y,u)\|_2 &\leq L_f\|x-y\|_2 && \forall x, y \in X, u \in U, \\
\|f(x,u)\|_2 &\leq M_f, && \forall x \in X, u \in U \ .
\end{aligned}$$

- $(H2)$ $\ell : X \times U \to \mathbb{R}$ is bounded and Lipschitz continuous with respect to x, meaning that there exist $L_\ell > 0$ and $M_\ell > 0$ such that

$$\begin{aligned}
|\ell(x,u) - \ell(y,u)| &\leq L_\ell\|x-y\|_2 && \forall x, y \in X, u \in U, \\
|\ell(x,u)| &\leq M_\ell, && \forall x \in X, u \in U \ .
\end{aligned}$$

5.3.1. *Approximation of $S^\delta w$.*

Lemma 18. *Let X be a convex subset of $\mathbb{R}^n$. We make assumptions* (H1) *and* (H2). *Let $w : x \to \mathbb{R}$ be such that w is $\mathcal{C}^1$ on a neighborhood of X, Lipschitz continuous with Lipschitz constant L_w with respect to the euclidean norm, c_1-semiconvex and c_2-semiconcave. Then there exists $K_1 > 0$ such that $\|[S^\delta w]_H - S^\delta w\|_\infty \leq K_1\delta^2$, for $\delta > 0$, where $[S^\delta w]_H$ is given by* (20).

Proof. We first show that there exists $K_1 > 0$ such that

$$[S^\delta w]_H(x) - S^\delta w(x) \geq -K_1 \delta^2, \quad \forall x \in X \ .$$

For all $x \in X$ and $u \in U$, define $\mathbf{x}_{u,x}$ to be the trajectory such that $\dot{\mathbf{x}}_{u,x}(s) = f(\mathbf{x}_{u,x}(s), u)$ and $\mathbf{x}_{u,x}(0) = x$. In other words, we apply a constant control u. We have

$$(S^\delta w)(x) \geq \sup\{ \int_0^\delta \ell(\mathbf{x}_{u,x}(s), u) ds + w(\mathbf{x}_{u,x}(\delta)) \mid u \in U\} \ . \ .$$

Since ℓ is Lipschitz continuous and f is bounded, we have

$$\begin{aligned} \Big| \int_0^\delta [\ell(\mathbf{x}_{u,x}(s), u) - \ell(x,u)] ds \Big| &\leq L_\ell \int_0^\delta \|\mathbf{x}_{u,x}(s) - x\|_2 ds \\ &\leq L_\ell \int_0^\delta M_f s ds \ , \end{aligned}$$

then

$$\Big| \int_0^\delta [\ell(\mathbf{x}_{u,x}(s), u) - \ell(x,u)] ds \Big| \leq \frac{1}{2} L_\ell M_f \delta^2 \ . \tag{31}$$

Therefore

$$(S^\delta w)(x) \geq -\frac{1}{2} L_\ell M_f \delta^2 + \sup\{\delta \ell(x,u) + w(\mathbf{x}_{u,x}(\delta)) \mid u \in U\} \ .$$

Since w is Lipschitz continuous and f is bounded and Lipschitz continuous, we have

$$\begin{aligned} \Big| w(\mathbf{x}_{u,x}(\delta)) - w(x + \delta f(x,u)) \Big| &\leq L_w \|\mathbf{x}_{u,x}(\delta) - x - \delta f(x,u)\|_2 \\ &\leq L_w \int_0^\delta \|f(\mathbf{x}_{u,x}(s), u) - f(x,u)\|_2 ds \\ &\leq L_w \int_0^\delta L_f \|\mathbf{x}_{u,x}(s) - x\|_2 ds \\ &\leq L_w L_f \int_0^\delta M_f s ds \ , \end{aligned}$$

and so

$$\Big| w(\mathbf{x}_{u,x}(\delta)) - w(x + \delta f(x,u)) \Big| \leq \frac{1}{2} L_w L_f M_f \delta^2 \ . \tag{32}$$

Moreover, since w is c_1-semiconvex, we have

$$w(x + \delta f(x,u)) \geq w(x) + \delta \nabla w(x) \cdot f(x,u) - \frac{c_1}{2} M_f^2 \delta^2 \ . \tag{33}$$

We deduce from (31), (32) and (33)

$$\begin{aligned} (S^\delta w)(x) &\geq -\big(L_\ell M_f + L_w L_f M_f + c_1 M_f^2\big) \frac{\delta^2}{2} + w(x) \\ &\qquad + \sup_{u \in U} \big\{ \delta \ell(x,u) + \delta \nabla w(x) \cdot f(x,u) \big\} \\ &\geq -\big(L_\ell M_f + L_w L_f M_f + c_1 M_f^2\big) \frac{\delta^2}{2} + w(x) + \delta H(x, \nabla w(x)) \ . \end{aligned}$$

This ends the first part of the proof.

We now prove an opposite inequality. For all $x \in X$ and for all measurable functions $\mathbf{u} : [0,\delta] \to U$, define $\mathbf{x}_{\mathbf{u},x}$ to be the trajectory such that $\dot{\mathbf{x}}_{\mathbf{u},x}(s) =$

$f(\mathbf{x}_{\mathbf{u},x}(s),\mathbf{u}(s))$ and $\mathbf{x}_{\mathbf{u},x}(0)=x$. Since $\ell(x,u)\leq H(x,p)-p\cdot f(x,u)$, for all $p\in\mathbb{R}^n$, $x\in X$ and $u\in U$, we deduce that

$$\begin{aligned}(S^\delta w)(x)\leq \sup\Big\{\int_0^\delta H(\mathbf{x}_{\mathbf{u},x}(s),\nabla w(x))ds+w(\mathbf{x}_{\mathbf{u},x}(\delta))\\ -\nabla w(x)\cdot\int_0^\delta f(\mathbf{x}_{\mathbf{u},x}(s),\mathbf{u}(s))ds\mid \mathbf{u}:[0,\delta]\to U\Big\}\\ =\sup\Big\{\int_0^\delta H(\mathbf{x}_{\mathbf{u},x}(s),\nabla w(x))ds\\ +w(\mathbf{x}(\delta))-\nabla w(x)\cdot\big(\mathbf{x}_{\mathbf{u},x}(\delta)-x\big)\mid \mathbf{u}:[0,\delta]\to U\Big\}\ .\end{aligned}$$

Using the fact that ℓ and f are Lipschitz continuous with respect to x, we have for all $x,x'\in X$, $p\in\mathbb{R}^n$

$$\Big|H(x,p)-H(x',p)\Big|\leq\big(L_\ell+L_f\|p\|_2\big)\|x-x'\|_2\ ,$$

therefore

$$\begin{aligned}(S^\delta w)(x)\quad&\leq\quad \sup\Big\{\big(L_\ell+L_fL_w\big)\int_0^\delta\|\mathbf{x}_{\mathbf{u},x}(s)-x\|_2ds+\delta H(x,\nabla w(x))\\ &\qquad +w(\mathbf{x}_{\mathbf{u},x}(\delta))-\nabla w(x)\cdot\big(\mathbf{x}_{\mathbf{u},x}(\delta)-x\big)\mid \mathbf{u}:[0,\delta]\to U\Big\}\\ &\leq\quad \big(L_\ell+L_fL_w\big)M_f\frac{\delta^2}{2}+\delta H(x,\nabla w(x))\\ &\qquad +\sup\Big\{w(\mathbf{x}_{\mathbf{u},x}(\delta))-\nabla w(x)\cdot\big(\mathbf{x}_{\mathbf{u},x}(\delta)-x\big)\mid \mathbf{u}:[0,\delta]\to U\Big\}\ .\end{aligned}$$

Since w is c_2-semiconcave, we have

$$w(\mathbf{x}_{\mathbf{u},x}(\delta))\leq w(x)+\nabla w(x)\cdot\big(\mathbf{x}_{\mathbf{u},x}(\delta)-x\big)+\frac{c_2}{2}M_f^2\delta^2\ .$$

We obtain

$$(S^\delta w)(x)\leq\big(L_\ell+L_fM_{Dw}+c_2M_f\big)M_f\frac{\delta^2}{2}+w(x)+\delta H(x,\nabla w(x))\ .$$

To end the proof, we take $K_1=\frac{1}{2}\big(L_\ell M_f+L_fM_{Dw}M_f+\max(c_1,c_2)M_f^2\big)$. □

5.3.2. *Approximation of the matrix K_h by the matrix $\tilde{K}_H$.*

Lemma 19. *Let X be a compact subset of $\mathbb{R}^n$. We consider an upper semicontinuous function $\varphi:X\to\mathbb{R}$ and a Lipschitz continuous function $\psi:X\to\mathbb{R}$ with Lipschitz constant L_ψ with respect to a norm $\|\cdot\|$. For $\varepsilon\geq 0$, we define:*

$$F_\varepsilon=\{x\in X\mid\ \varphi(x)\geq\sup_{x'\in X}\varphi(x')-\varepsilon\},\tag{34a}$$

$$g(\varepsilon)=\sup_{x\in F_\varepsilon}d(x,F_0)\ ,\tag{34b}$$

where $d(x,F_0)=\inf_{y\in F_0}\|y-x\|$. We have:

$$|\sup_{x\in X}\big(\varphi(x)+\delta\psi(x)\big)-\big[\sup_{x\in X}\varphi(x)+\delta\sup_{x\in\arg\max\varphi}\psi(x)\big]\,|\leq L_\psi\delta g(\delta M)\ ,$$

where $M=\sup_{x\in X}\psi(x)-\inf_{x\in X}\psi(x)$.

Proof. Since φ is u.s.c. and X is compact, $F_0 = \arg\max \varphi$ and

$$\sup_{x\in X} \big(\varphi(x) + \delta\psi(x)\big) \geq \sup_{x\in X} \varphi(x) + \delta \sup_{x\in F_0} \psi(x) \ . \tag{35}$$

For $\varepsilon > 0$ we have:

$$\sup_{x\in X} \big(\varphi(x) + \delta\psi(x)\big) = \max\Big[\sup_{x\in F_\varepsilon} \big(\varphi(x) + \delta\psi(x)\big), \sup_{x\in X\setminus F_\varepsilon} \big(\varphi(x) + \delta\psi(x)\big)\Big] \ .$$

Let $\varepsilon = \delta(\sup_{x\in X} \psi(x) - \inf_{x\in X} \psi(x)) = M\delta$ (which is finite since ψ is continuous and X is compact). We have:

$$\begin{aligned}
\sup_{x\in X\setminus F_\varepsilon} \big(\varphi(x) + \delta\psi(x)\big) &\leq -\varepsilon + \sup_{x\in X} \varphi(x) + \delta \sup_{x\in X} \psi(x) \\
&= \sup_{x\in F_\varepsilon} \varphi(x) + \delta \inf_{x\in X} \psi(x) \\
&\leq \sup_{x\in F_\varepsilon} \big[\varphi(x) + \delta\psi(x)\big] \ .
\end{aligned}$$

Therefore

$$\begin{aligned}
\sup_{x\in X} \big(\varphi(x) + \delta\psi(x)\big) &= \sup_{x\in F_\varepsilon} \big(\varphi(x) + \delta\psi(x)\big) \\
&\leq \sup_{x\in X} \varphi(x) + \delta \sup_{x\in F_\varepsilon} \psi(x) \ . \qquad (36)
\end{aligned}$$

We deduce from (35) and (36):

$$0 \leq \sup_{x\in X} \big(\varphi(x) + \delta\psi(x)\big) - \Big[\sup_{x\in X} \varphi(x) + \delta \sup_{x\in F_0} \psi(x)\Big] \leq \delta\Big[\sup_{x\in F_\varepsilon} \psi(x) - \sup_{x\in F_0} \psi(x)\Big] \ .$$

Since ψ is Lipschitz continuous, we have

$$\begin{aligned}
\sup_{x\in F_\varepsilon} \psi(x) - \sup_{x\in F_0} \psi(x) &= \sup_{x\in F_\varepsilon} \inf_{y\in F_0} \big(\psi(x) - \psi(y)\big) \\
&\leq \sup_{x\in F_\varepsilon} \inf_{y\in F_0} L_\psi \|x - y\| \\
&= L_\psi \sup_{x\in F_\varepsilon} d(x, F_0) \\
&= L_\psi g(\varepsilon) \ .
\end{aligned}$$

□

Corollary 20. *Let X be a compact convex subset of $\mathbb{R}^n$. We consider an u.s.c. and strongly concave function $\varphi : X \to \mathbb{R}$ with modulus $c > 0$ and a Lipschitz continuous function $\psi : X \to \mathbb{R}$ with Lipschitz constant L_ψ with respect to the euclidean norm. Then the maximum of φ on X is attained at a unique point $x_0 \in X$ i.e. $\arg\max_X \varphi = \{x_0\}$ and*

$$|\sup_{x\in X} \big(\varphi(x) + \delta\psi(x)\big) - \big(\varphi(x_0) + \delta\psi(x_0)\big)| \leq L_\psi \delta \sqrt{\frac{2\delta M}{c}} \ ,$$

where $M = \sup_{x\in X} \psi(x) - \inf_{x\in X} \psi(x)$.

Proof. Define $\Phi(x) = \varphi(x_0) - \varphi(x)$ for $x \in X$ and $\Phi(x) = +\infty$ elsewhere. We have $\Phi(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $\Phi(x_0) = 0$. Since Φ is l.s.c. and convex on $\mathbb{R}^n$, then $0 \in \partial\Phi(x_0)$. Moreover Φ is strongly convex with modulus c. Then, using Theorem 6.1.2 of [HUL93, Chapter VI] we have for all $x, x' \in X$

$$\Phi(x) \geq \Phi(x') + \langle s \mid x - x'\rangle + \frac{c}{2}\|x - x'\|_2^2 \quad \forall s \in \partial\Phi(x') \ .$$

Taking $x' = x_0$ and $s = 0$ we obtain for all $x \in X$

$$\Phi(x) \geq \frac{c}{2}\|x - x_0\|_2^2 \ ,$$

which implies

$$\varphi(x) \leq \varphi(x_0) - \frac{c}{2}\|x - x_0\|_2^2 \quad \forall x \in X \ .$$

Using the notations of the previous Lemma, we get easily (see also Proposition 4.32 of [BS00]) for all $x \in F_\varepsilon$, $d(x, F_0) \leq \sqrt{\frac{2\varepsilon}{c}}$, where $\varepsilon = \delta\big(\sup_{x\in X}\psi(x) - \inf_{x\in X}\psi(x)\big)$. $\square$

Remark 21. To have an error estimate of the approximation of the matrix $K_{H,h}$ by the matrix $\tilde{K}_{H,h}$, we apply Lemma 20 in the case where

$$\varphi(x) = w_i(x) + z_j(x) \quad \text{and} \quad \psi(x) = H(x, \nabla w_i(x)) \ ,$$

for a suitable choice of the finite elements w_i and test functions z_j. Using Assumptions (H_1) and (H_2), we have that, for all $x \in X$, $|\psi(x)| \leq M_f\|\nabla w\|_\infty + M_\ell$, where $\|\nabla w\|_\infty = \|\|\nabla w\|_2\|_\infty$ and $\nabla w = (\nabla w_i)_{1\leq i\leq p}$. We deduce

$$\sup\psi - \inf\psi \leq 2\big(M_f\|\nabla w\|_\infty + M_\ell\big) \ .$$

Moreover $H(\cdot, p)$ and $H(x, \cdot)$ are Lipschitz continuous with Lipschitz constants $L_f\|p\|_2 + L_\ell$ and M_f respectively. Hence, ψ is Lipschitz continuous with Lipschitz constant

$$L_\psi = L_f\|\nabla w\|_\infty + L_\ell + M_f\|D^2 w_i\|_\infty \ .$$

5.4. **Final estimation of the error of the MFEM.** We now state our main convergence result, which holds for quadratic finite elements and Lipschitz test functions.

Theorem 22. *Let X be a compact convex subset of $\mathbb{R}^n$ with non-empty interior and $\hat{X} = X + \mathrm{B}_2(0, \frac{L}{c})$, where $L > 0$, $c > 0$. Choose any finite sets of discretization points $\mathcal{T} \subset \mathbb{R}^n$ and $\hat{\mathcal{T}} \subset \mathbb{R}^n$. Let*

$$\Delta x = \max(\rho_X(\mathcal{T}), \rho_{\hat{X}}(\hat{\mathcal{T}})).$$

We make assumptions (H1) *and* (H2)*, and assume that the value function at time t, v^t, is c-semiconvex and Lipschitz continuous with constant L with respect to the euclidean norm, for all $t \geq 0$. Let us choose quadratic finite elements $w_{\hat{x}}$ of Hessian c, centered at the points $\hat{x}$ of $\hat{\mathcal{T}}$. Let us choose, as test functions, the Lipschitz finite elements $z_{\hat{y}}$ with constant $a \geq L$, centered at the points $\hat{y}$ of $\mathcal{T}$. For $t = 0, \delta, \ldots, T$, let v_h^t be the approximation of v^t given by the max-plus finite element method implemented with the approximation $K_{H,h}$ of K_h given by* (21)*. Then, there exists a constant $C_1 > 0$ such that*

$$\|v_h^T - v^T\|_\infty \leq C_1(\delta + \frac{\Delta x}{\delta}) \ .$$

When the approximation $K_{H,h}$ is replaced by $\tilde{K}_{H,h}$, given by (22)*, this inequality becomes:*

$$\|v_h^T - v^T\|_\infty \leq C_2(\sqrt{\delta} + \frac{\Delta x}{\delta}) \ ,$$

for some constant $C_2 > 0$.

Proof. Let $\mathcal{W}_h$ and $\mathcal{Z}_h$ denote the complete semimodules of $\overline{\mathbb{R}}_{\max}^X$ generated by the families $(w_{\hat{x}})_{\hat{x}\in\hat{\mathcal{T}}}$ and $(z_{\hat{y}})_{\hat{y}\in\mathcal{T}}$ respectively. We index the elements of $\hat{\mathcal{T}}$ and $\mathcal{T}$ by $\hat{x}_1,\cdots,\hat{x}_p$ and $\hat{y}_1,\cdots,\hat{y}_q$ respectively. Using Corollary 12, we have

$$\begin{aligned}\|v_h^T - v^T\|_\infty \quad \leq \quad & (1+\frac{T}{\delta})\Big(\sup_{t\in\bar{\tau}_\delta}\big(\|P^{-\mathcal{Z}_h}(v^t)-v^t\|_\infty + \|P_{\mathcal{W}_h}(v^t)-v^t\|_\infty\big) \\ & + \max_{1\leq i\leq p}\|[S^\delta w_i]_H - S^\delta w_i\|_\infty\Big) \ .\end{aligned}$$

To estimate the projection error $\|P_{\mathcal{W}_h}(v^t)-v^t\|_\infty$, we apply Lemma 16 for $\hat{X}_h=\hat{\mathcal{T}}$. We obtain, for $t\in\bar{\tau}_\delta$, $\|P_{\mathcal{W}_h}(v^t)-v^t\|_\infty \leq c\operatorname{diam}X\Delta x$. Applying Lemma 17 we obtain, for $t\in\bar{\tau}_\delta$, $\|P^{-\mathcal{Z}_h}(v^t)-v^t\|_\infty\leq n(a+L)\Delta x$. Finally, using Lemma 18, we get

$$\|v_h^T - v^T\|_\infty \leq C_1(\delta+\frac{\Delta x}{\delta}) \ ,$$

where

$$C_1 > (T+1)\max\Big(c\operatorname{diam}X + n(a+L), \frac{1}{2}(L_\ell M_f + L_f M_f(\operatorname{diam}X+\frac{L}{c}) + cM_f^2)\Big).$$

To prove the second inequality, we use Lemma 13 together with Remark 21. Using the notation of Corollary 20 and the fact that $\varphi = w_i + z_j$ is c-strongly convex, we have $\sup\psi - \inf\psi \leq 2(M_\ell + M_f c(\operatorname{diam}X+\frac{L}{c}))$ and $L_\psi = L_\ell + cM_f + L_f c(\operatorname{diam}X+\frac{L}{c})$. We deduce that

$$|(\tilde{K}_{H,h})_{ji} - (K_{H,h})_{ji}| \leq 2\big(L_\ell + cM_f + L_f c(\operatorname{diam}X+\frac{L}{c})\big)\sqrt{\frac{M_\ell}{c}+M_f(\operatorname{diam}X+\frac{L}{c})}\delta\sqrt{\delta} \ ,$$

for $i=1,\cdots,p$ and $j=1,\cdots,q$. Hence, there exists $C2>0$ such that

$$\|v_h^T - v^T\|_\infty \leq C_2(\sqrt{\delta}+\frac{\Delta x}{\delta}) \ ,$$

when δ is small enough. □

A variant of this theorem, with a stronger assumption, was proved in [Lak03].

Remark 23. When $\mathcal{T}$ is a rectangular grid of step $h>0$, meaning that $\mathcal{T}$ is the intersection of $(\mathbb{Z}h)^n$ with a cartesian product of bounded intervals, we have

$$\rho_X(\mathcal{T}) \leq \sqrt{n}h.$$

Hence, when $\mathcal{T}$ and $\hat{\mathcal{T}}$ are both rectangular grids of step h, we have $\Delta x \leq \sqrt{n}h = O(h)$ in Theorem 22.

6. Numerical results

This section presents the results of numerical experiments with the MFEM described in Section 3. We consider optimal control problems in dimension 1 and 2 whose value functions are known or can be computed by solving the Riccati equation (in the case of linear quadratic problems).

6.1. **Implementation.** We implemented the MFEM using the max-plus toolbox of Scilab [Plu98] (in dimension 1) and specific programs written in C (in dimension 2). We used the approximation $\tilde{K}_{H,h}$ of the matrix K_h. The matrix M_h can always be computed analytically. In all the examples below, the Hamiltonian H, and so the stiffness matrix $\tilde{K}_{H,h}$, have been computed analytically. We avoided storing the (full) matrices M_h and $\tilde{K}_{H,h}$ when the number of discretization points is large.

6.2. **Examples in dimension1.** The next two examples are inspired by those proposed by M. Falcone in [BCD97].

Example 24. We consider the case where $T = 1$, $\phi \equiv 0$, $X = [-1, 1]$, $U = [0, 1]$, $\ell(x, u) = x$ and $f(x, u) = -xu$. Assumptions $(H1)$ and $(H2)$ are satisfyied. The optimal choice is to take $u^* = 0$ whenever $x > 0$ and to move on the right with maximum speed ($u^* = 1$) whenever $x \leq 0$. For all $t \in [0, T]$, the value function is:

$$v(x, t) = \begin{cases} xt & \text{if } x > 0 \\ x(1 - e^{-t}) & \text{otherwise.} \end{cases}$$

We choose quadratic finite elements w_i of Hessian c centered at the points of the regular grid $(\mathbb{Z}\Delta x) \cap [-2, 2]$ and Lipschitz finite elements z_j with constant $a \geq 1$ centered at the points of the regular grid $(\mathbb{Z}\Delta x) \cap X$. We represent in Figure 3 the solution given by our algorithm in the case where $\delta = 0.01$, $\Delta x = 0.005$, $a = 1.5$ and $c = 1$. We obtain a L_∞-error of order 10^{-2}.

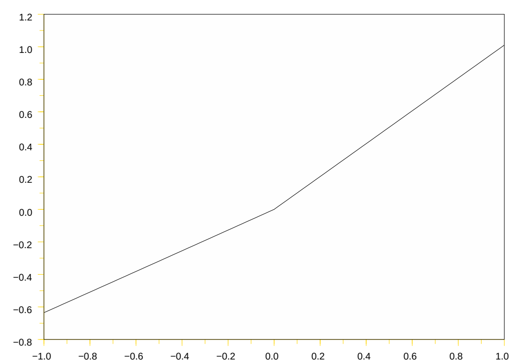

FIGURE 3. Max-plus approximation (Example 24)

Example 25. We consider the case where $T = 1$, $\Phi \equiv 0$, $X = [-1, 1]$, $U = [-1, 1]$, $\ell(x, u) = -3(1 - |x|)$ and $f(x, u) = u(1 - |x|)$. It is clear that ℓ and f are bounded and Lipschitz continuous functions. The optimal choice is to take $u^* = -1$ whenever $x > 0$ and $u^* = 1$ whenever $x < 0$. Therefore, all the trajectories lie in X. For all $t \in [0, T]$, the value function is:

$$v(x, t) = -3(1 - |x|)(1 - e^{-t})$$

We choose quadratic finite elements w_i of Hessian c and Lipschitz finite elements z_j with constant a. We represent in Figure 4 the solution given by our algorithm in the case where $\delta = 0.02$, $\Delta x = 0.01$, $a = 2$ and $c = 8$. We obtain a L_∞-error of order $7.66 \cdot 10^{-3}$.

Example 26 (Linear Quadratic Problem)**.** We consider the case where $U = \mathbb{R}$, $X = \mathbb{R}$,

$$\ell(x, u) = -\frac{1}{2}(x^2 + u^2), \quad f(x, u) = u, \text{ and } \phi \equiv 0 \ .$$

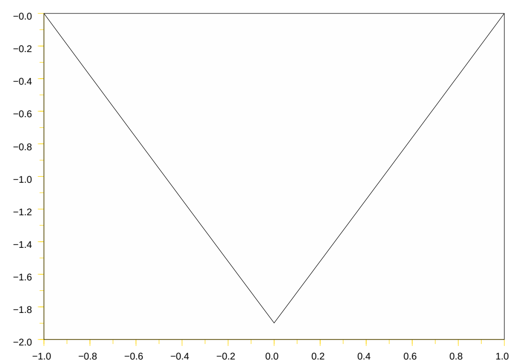

FIGURE 4. Max-plus approximation (Example 25)

The Hamiltonian is $H(x,p) = -\frac{x^2}{2} + \frac{p^2}{2}$. This problem can be solved analytically. For $x \in X$, the value function at time t is

$$v(x,t) = -\frac{1}{2}\tanh(t)x^2.$$

The domain X is unbounded and ℓ and f are unbounded and locally Lipschitz continuous. We will restrict X to the set $[-5;5]$ so that ℓ and f satisfy Assumptions $(H1)$ and $(H2)$.
We choose quadratic finite elements w_i and z_j of Hessian $c=1$, centered at the points of the regular grid $(\mathbb{Z}\Delta x) \cap [-6,6]$. We represent in Figure 5 the solution given by our algorithm in the interval $[-1;1]$ in the case where $T=5$, $\delta = 0.5$, $\Delta x = 0.05$ and $L=1$. We obtain a L_∞-error of $4.54 \cdot 10^{-5}$.

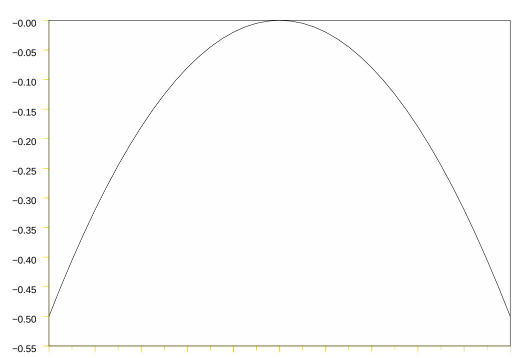

FIGURE 5. Max-plus approximation of a linear quadratic control problem (Example 26)

Example 27 (Distance problem)**.** We consider the case where $T = 1$, $\phi \equiv 0$, $X = [-1, 1]$, $U = [-1, 1]$,

$$\ell(x,u) = \begin{cases} -1 & \text{if} \quad x \in (-1,1), \\ 0 & \text{if} \quad x \in \{-1,1\}, \end{cases} \quad \text{and} \quad f(x,u) = \begin{cases} u & \text{if} \quad x \in (-1,1), \\ 0 & \text{if} \quad x \in \{-1,1\}. \end{cases}$$

Putting $\ell = 0$ and $f = 0$ on ∂X keeps the trajectories in the domain X but we loose the Lipschitz continuity of ℓ and f. For $x \in X$, the value function at time t of this problem is

$$v(x,t) = \max(-t, |x| - 1).$$

Consider first quadratic finite elements w_i and z_j of Hessian c, centered at the points of the regular grid $(\mathbb{Z}\Delta x) \cap \big(X + \mathrm{B}_\infty(0, \frac{L}{c})\big)$. In Figure 6, we represent the solution given by our algorithm in the case where $\delta = 0.02$, $\Delta x = 0.01$, $c = 2$ and $L = 1$. Since $\Pi^{Z_h^*}$ is a projector on a subsemimodule of the $\overline{\mathbb{R}}_{\min}$-semimodule of c-semiconcave functions, and since the solution is not c-semiconcave for any c, the error of projection $\|\Pi^{Z_h^*}(v^t) - v^t\|_\infty$ does not converge to zero when Δx goes to zero, which explains the magnitude of the error.

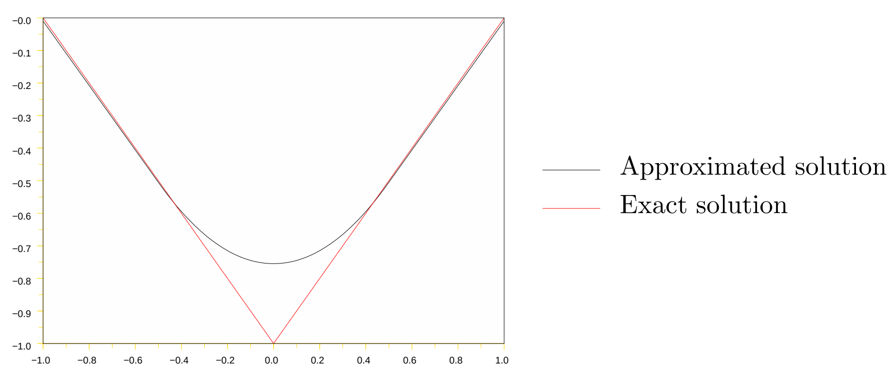

FIGURE 6. A bad choice of test functions for the distance problem (Example 27)

To solve this problem, it suffices to replace the test functions z_j by the Lipschitz finite elements with constant $a \geq 1$, centered at the points of the regular grid $(\mathbb{Z}\Delta x) \cap [-1, 1]$. This is illustrated in Figure 7 in the case where $\delta = 0.02$, $\Delta x = 0.01$, $c = 2$ and $a = 1.1$. We obtain a L_∞-error of $1.05 \cdot 10^{-2}$.

6.3. **Examples in dimension 2.**

Example 28 (Linear Quadratic Problem in dimension 2)**.** We consider the case where $U = \mathbb{R}^2$, $X = \mathbb{R}^2$, $\phi \equiv 0$,

$$\ell(x,u) = -\frac{x_1^2 + x_2^2}{2} - \frac{u_1^2 + u_2^2}{2} \quad \text{and} \quad f(x,u) = u \ .$$

For $x \in X$, the value functions at time t is

$$v(x,t) = -\frac{1}{2}\tanh(t)(x_1^2 + x_2^2).$$

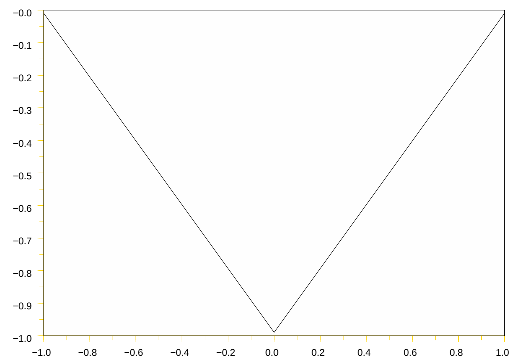

FIGURE 7. A good choice of test functions for the distance problem (Example 27)

As in Example 26, the domain X is unbounded therefore ℓ and f do not satisfy Assumptions $(H1)$ and $(H2)$. We will restrict the domain to the set $[-5;5]^2$.
We choose quadratic finite elements w_i and z_j of Hessian c centered at the points of the regular grid $\big((\mathbb{Z}\Delta x) \cap [-6,6]\big)^2$. We represent in Figure 8 the solution given by our algorithm in the case where $T = 5$, $\delta = 0.5$, $\Delta x = 0.1$, $c = 1$. The L_∞-error

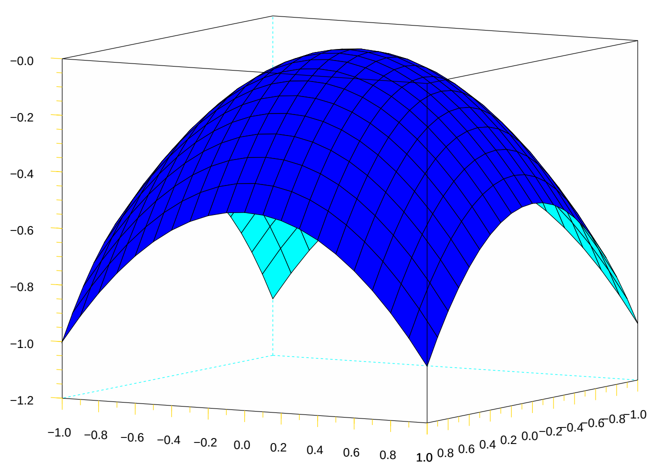

FIGURE 8. Max-plus approximation of a linear quadratic control problem (Example 28)

is $9 \cdot 10^{-5}$.

Example 29 (Distance problem in dimension 2)**.** We consider the case where $T = 1$, $\phi \equiv 0$, $X = [-1,1]^2$, $U = [-1,1]^2$,

$$\ell(x,u) = \begin{cases} -1 & \text{if} \quad x \in \operatorname{int} X, \\ 0 & \text{if} \quad x \in \partial X, \end{cases}$$

$$f(x,u) = \begin{cases} u & \text{if} \quad x \in \operatorname{int} X, \\ 0 & \text{if} \quad x \in \partial X. \end{cases}$$

For $x \in X$, the value function at time t is

$$v(x,t) = \max\big(-t, \max(|x_1|,|x_2|) - 1\big).$$

We choose quadratic finite elements w_i of Hessian c centered at the points of the regular grid $\big((\mathbb{Z}\Delta x) \cap [-3,3]\big)^2$ and Lipschitz finite elements z_j with constant a centered at the points of the regular grid $\big((\mathbb{Z}\Delta x) \cap [-1,1]\big)^2$. We represent in Figure 9 the solution given by our algorithm in the case where $T = 1$, $\delta = 0.05$, $\Delta x = 0.025$, $a = 3$ and $c = 1$. The L_∞-error is of order 0.05.

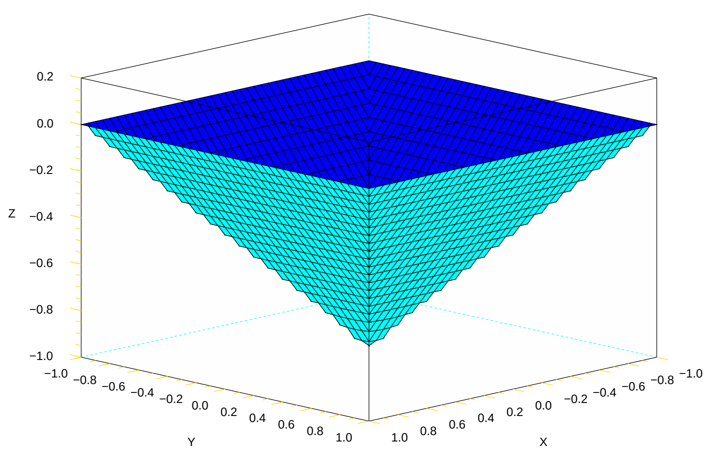

FIGURE 9. Max-plus approximation of the distance problem (Example 29)

Example 30 (Rotating problem)**.** We consider here the Mayer problem where $T = 1$, $X = \mathrm{B}_2(0,1)$, $U = \{0\}$, $\phi(x) = -\frac{1}{2}x_1^2 - \frac{3}{2}x_2^2$, $\ell(x,u) = 0$ and $f(x,u) = (-x_2, x_1)$. For $x \in X$, the value function at time t is

$$v(x,t) = -\frac{1}{2}(-x_2 \sin(t) + x_1 \cos(t))^2 - \frac{3}{2}(x_2 \cos(t) + x_1 \sin(t))^2.$$

We choose quadratic finite elements w_i and z_j of Hessians c_w and c_z respectively, centered at the points of the regular grid $\big((\mathbb{Z}\Delta x) \cap [-2,2]\big)^2$. We represent in Figure 10 the solution given by our algorithm in the case where $\delta = \Delta x = 0.05$, $c_w = 4$ and $c_z = 3$. The L_∞-error is 0.046.

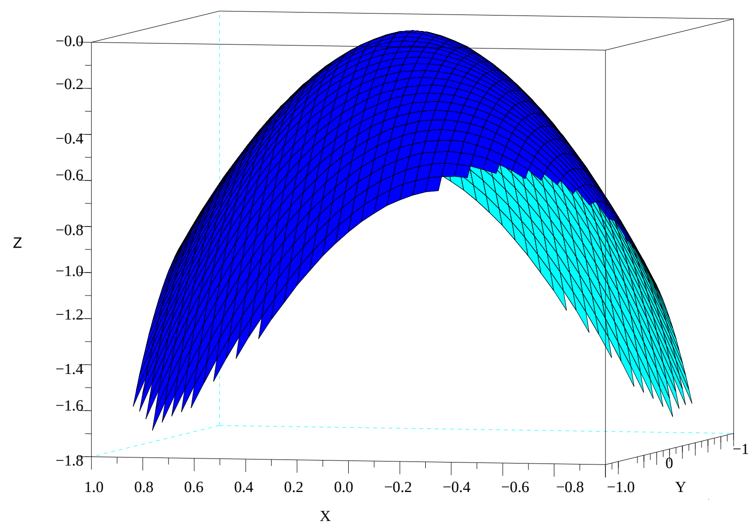

FIGURE 10. Max-plus approximation of the rotating problem (Example 30)

Example 31. We consider the case where $U = \mathbb{R}$, $X = \mathbb{R}^2$, $\phi(x) = -x_1^2 - 2x_2^2$,

$$\ell(x,u) = -x_1^2 - \frac{u^2}{2} \quad \text{and} \qquad f(x,u) = (x_2, u)^T \ .$$

We choose quadratic finite elements w_i and z_j of Hessian c_w and c_z respectively centered at the points of the grids $\big((\mathbb{Z}\Delta x) \cap [-2,2]\big)^2$ and $\big((\mathbb{Z}\Delta x) \cap [-11,11]\big)^2$ respectively. We represent in Figure 11 the solution given by our algorithm in the case where $T = 1$, $\delta = 0.05$, $\Delta x = 0.025$, $c_w = 10$ and $c_z = 1$. The L_∞-error is 0.11. (We compared the max-plus approximation with the solution of the problem given by the Riccati equation).

6.4. **Conclusion.** We have tested our method on examples that fullfill the assumptions of Theorem 22 (see Examples 24, 25, 30) but also on problems that do not fullfill these assumptions. The method is efficient even in the second case. The only difficulty comes from the full character of the matrices M_h and K_h, which limits the number of discretization points. To treat higher dimensional examples, we need higher order approximations (when the value function is regular enough). This is the object of a subsequent work.

REFERENCES

[AGL04] M. Akian, S. Gaubert, and A. Lakhoua. A max-plus finite element method for solving finite horizon deterministic optimal control problems. In *Proceedings of the Sixteenth International Symposium on Mathematical Theory of Networks and Systems (MTNS'04)*. Leuven, Belgium, 2004. And arXiv:math.OC/0404184, April 2004.

[Bar94] G. Barles. *Solutions de viscosité des équations de Hamilton-Jacobi*. Springer Verlag, 1994.

[BCD97] M. Bardi and I. Capuzzo-Dolcetta. *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*. Birkhaüser, 1997.

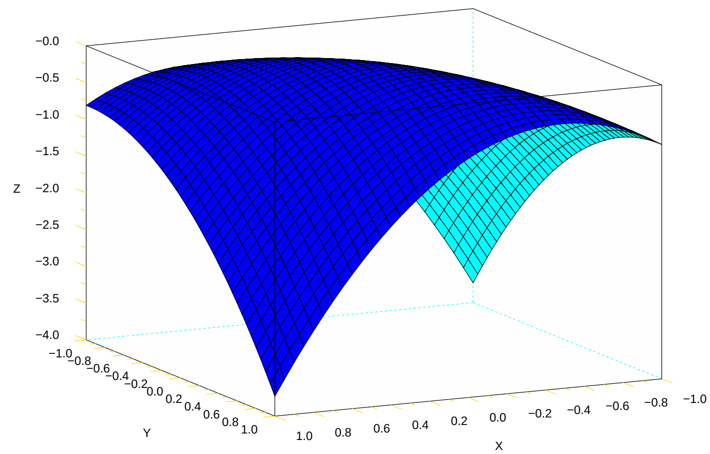

FIGURE 11. Max-plus approximation of the solution of the control problem of Example 31

[BCOQ92] F. Baccelli, G. Cohen, G. J. Olsder, and J.-P. Quadrat. *Synchronization and linearity : an algebra for discrete events systems*. John Wiley & Sons, New-York, 1992.

[BD99] M. Boué and P. Dupuis. Markov chain approximations for deterministic control problems with affine dynamics and quadratic cost in the control. *SIAM J. Numer. Anal.*, 36(3):667–695 (electronic), 1999. ISSN 0036-1429.

[Bir67] G. Birkhoff. *Lattice Theory*, volume 25. American Mathematical Society, 1967.

[BJ72] T. S. Blyth and M. F. Janowitz. *Residuation theory*. Pergamon Press, Oxford, 1972. International Series of Monographs in Pure and Applied Mathematics, Vol. 102.

[BS00] J. F. Bonnans and A. Shapiro. *Perturbation analysis of optimization problems*. Springer Series in Operations Research. Springer Verlag, New York, 2000.

[BZ05] O. Bokanowski and H. Zidani. Anti-dissipative schemes for advection and application to hamilton-jacobi-bellman equations. *J. Sci. Compt*, to appear 2005.

[CD83] I. Capuzzo Dolcetta. On a discrete approximation of the Hamilton-Jacobi equation of dynamic programming. *Appl. Math. Optim.*, 10(4):367–377, 1983.

[CDF89] I. Capuzzo-Dolcetta and M. Falcone. Discrete dynamic programming and viscosity solutions of the Bellman equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 6(suppl.):161–183, 1989. Analyse non linéaire (Perpignan, 1987).

[CDI84] I. Capuzzo-Dolcetta and H. Ishii. Approximate solutions of the Bellman equation of deterministic control theory. *Appl. Math. Optim.*, 11(2):161–181, 1984.

[CFF04] E. Carlini, M. Falcone, and R. Ferretti. An efficient algorithm for Hamilton-Jacobi equations in high dimension. *Comput. Vis. Sci.*, 7(1):15–29, 2004.

[CG79] R. Cuninghame-Green. *Minimax Algebra*. Number 166 in Lecture notes in Economics and Mathematical Systems. Springer Verlag, 1979.

[CGQ96] G. Cohen, S. Gaubert, and J.-P. Quadrat. Kernels, images and projections in dioids. In *Proceedings of the International Workshop on Discrete Event Systems (WODES'96)*. IEE, Edinburgh, UK, 1996.

[CGQ04] G. Cohen, S. Gaubert, and J.-P. Quadrat. Duality and separation theorem in idempotent semimodules. *Linear Algebra and Appl.*, 379:395–422, 2004. Eprint doi:10.1016/j.laa.2003.08.010. Also arXiv:math.FA/0212294.

[CL84] M. G. Crandall and P.-L. Lions. Two approximations of solutions of Hamilton-Jacobi equations. *Math. Comp.*, 43(167):1–19, 1984.

[CM04] G. Collins and W. McEneaney. Min-plus eigenvector methods for nonlinear H_∞ problems with active control. In *Optimal control, stabilization and nonsmooth analysis*, volume 301 of *Lecture Notes in Control and Inform. Sci.*, pages 101–120. Springer, Berlin, 2004.

[CT80] M. G. Crandall and L. Tartar. Some relations between non expansive and order preserving maps. *Proceedings of the AMS*, 78(3):385–390, 1980.

[DJLC53] M. Dubreil-Jacotin, L. Lesieur, and R. Croisot. *Théorie des treillis des structures algébriques ordonnées et des treillis géométriques*. Gauthier-Villars, Paris, 1953.

[Fal87] M. Falcone. A numerical approach to the infinite horizon problem of deterministic control theory. *Appl. Math. Optim.*, 15(1):1–13, 1987. Corrigenda in *Appl. Math. Optim.*, 23:213–214, 1991.

[Fat06] A. Fathi. *Weak KAM theorem in Lagrangian dynamics*. Cambridge University Press, 2006. To appear.

[FF94] M. Falcone and R. Ferretti. Discrete time high-order schemes for viscosity solutions of Hamilton-Jacobi-Bellman equations. *Numer. Math.*, 67(3):315–344, 1994. ISSN 0029-599X.

[FG99] M. Falcone and T. Giorgi. An approximation scheme for evolutive Hamilton-Jacobi equations. In *Stochastic analysis, control, optimization and applications*, Systems Control Found. Appl., pages 289–303. Birkhäuser Boston, Boston, MA, 1999.

[FM00] W. H. Fleming and W. M. McEneaney. A max-plus-based algorithm for a Hamilton-Jacobi-Bellman equation of nonlinear filtering. *SIAM J. Control Optim.*, 38(3):683–710, 2000. Eprint doi:10.1137/S0363012998332433.

[FS93] W. H. Fleming and H. M. Soner. *Controlled Markov processes and viscosity solutions*. Springer Verlag, New-York, 1993.

[GM01] M. Gondran and M. Minoux. *Graphes, Dioïdes et semi-anneaux*. TEC & DOC, Paris, 2001.

[Gon96] M. Gondran. Analyse MINPLUS. *C. R. Acad. Sci. Paris Sér. I Math.*, 323(4):371–375, 1996. ISSN 0764-4442.

[GR85] R. Gonzalez and E. Rofman. On deterministic control problems: an approximation procedure for the optimal cost, part I and II. *SIAM J. Control Optim.*, 23(2):242–285, 1985.

[HUL93] J.-B. Hiriart-Urruty and C. Lemaréchal. *Convex analysis and minimization algorithms I*. Springer Verlag, 1993.

[KM88] V. N. Kolokoltsov and V. P. Maslov. The Cauchy problem for the homogeneous Bellman equation. *Soviet Math. Dokl.*, 36(2):326–330, 1988.

[KM97] V. N. Kolokoltsov and V. P. Maslov. *Idempotent analysis and applications*. Kluwer Acad. Publisher, 1997.

[Lak03] A. Lakhoua. *Résolution numérique de problèmes de commande optimale déterministe et algèbre max-plus*. Rapport de DEA, Université Paris VI, 2003.

[Lio82] P.-L. Lions. *Generalised solutions of Hamilton-Jacobi equations*. Pitman, 1982.

[LMS01] G. L. Litvinov, V. P. Maslov, and G. B. Shpiz. Idempotent functional analysis: an algebraic approach. *Math. Notes*, 69(5):696–729, 2001. Eprint doi:10.1023/A:1010266012029. Also arXiv:math.FA/0009128.

[Mas73] V. Maslov. *Méthodes Operatorielles*. Mir, Moscou, 1973. French Transl. 1987.

[McE02] W. M. McEneaney. Error analysis of a max-plus algorithm for a first-order HJB equation. In *Stochastic theory and control (Lawrence, KS, 2001)*, volume 280 of *Lecture Notes in Control and Inform. Sci.*, pages 335–351. Springer, Berlin, 2002.

[McE03] W. M. McEneaney. Max-plus eigenvector representations for solution of nonlinear H_∞ problems: basic concepts. *IEEE Trans. Automat. Control*, 48(7):1150–1163, 2003. ISSN 0018-9286.

[McE04] W. M. McEneaney. Max-plus eigenvector methods for nonlinear H_∞ problems: Error analysis. *SIAM J. Control Optim.*, 43(2):379–412 (electronic), 2004.

[MH98] W. M. McEneaney and M. Horton. Max-Plus eigenvector representations for nonlinear H_∞ value functions. In *Proceedings of the 37th Conference on Decision and Control (CDC'98)*, pages 3506–3511. IEEE, 1998.

[MH99] W. M. McEneaney and M. Horton. Computation of max-plus eigenvector representations for nonlinear H_∞ value functions. In *Americam Control Conference*, pages 1400–1404. 1999.

[MS92] V. P. Maslov and S. Samborskiĭ, editors. *Idempotent analysis*, volume 13 of *Adv. in Sov. Math.* AMS, RI, 1992.

[OS91] S. Osher and C.-W. Shu. High-order essentially nonoscillatory schemes for Hamilton-Jacobi equations. *SIAM J. Numer. Anal.*, 28(4):907–922, 1991.

[Plu98] M. Plus. Documentation of the max-plus toolbox of Scilab, 1998. Available from ftp://ftp.inria.fr/INRIA/Scilab/contrib/MAXPLUS/.

[Roc70] R. T. Rockafellar. *Convex analysis*. Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970.

[SU00] J.-R. Sack and J. Urrutia. *Handbook of computational geometry*. North-Holland, Amsterdam, 2000.

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